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Predictive studies of the Lotka-Volterra biocompetitive system based on transition theory

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Abstract: *With the rapid development of science and technology, the loss of biodiversity and the degradation of ecosystems are becoming increasingly important for human survival and development. This article adopts the theory of dynamic transition to investigate the spatial dynamics characteristics of the Lotka-Volterra competitive model with diffusion terms. It determines the new solution expressions for the system transitioning from zero to nonzero solutions and utilizes spectral theory of linear fully continuous fields to demonstrate continuous transitions occurring when the bifurcation parameter exceeds the stability threshold of the system. Finally, the theoretical findings are validated using finite difference method. The research results obtained in this paper have certain theoretical significance and application value, which can be applied in many practical aspects, providing a strong theoretical basis for the protection of ecological stability and diversity.*

Keywords: *Lotka-Volterra model in biological competition; dynamical transition theory; PES condition; finite difference method*

1. Introduction

The rapid development of China's economy and agriculture is accompanied by an accelerating global rate of species extinction due to anthropogenic factors. The loss of biodiversity and degradation of ecosystems pose significant threats to human survival and development. Effectively addressing this global biodiversity and ecosystem crisis has become an inevitable trend. Consequently, research on population dynamics and the dynamic transitions of population systems has emerged as an imperative factor.

In 1979, in order to study the spatial distribution of two competing species under intra- and interpopulation reproduction pressure, biologists Shigesada, Kawasaki, and Teramoto firstly proposed the S-K-T model of biological competition, abbreviated as S-K-T model. What they proposed in literature[14] is the following S-K-T model with cross-diffusion and self-diffusion terms, which can be expressed as follows.

$$\begin{cases} u_t = \Delta\left[\left(d_1+\rho_{11}u+\rho_{12}v\right)u\right]+u\left(a_1-b_1u-c_1v\right), & x\in\Omega, t>0 \\ v_t = \Delta\left[\left(d_2+\rho_{21}u+\rho_{22}v\right)v\right]+v\left(a_2-b_2u-c_2v\right), & x\in\Omega, t>0 \\ \dfrac{\partial u}{\partial \nu}=\dfrac{\partial v}{\partial \nu}=0, & x\in\partial\Omega, t>0 \end{cases} \tag{1}$$

In the above system of equations, Δ is the Laplace operator, u and v is the population density of two competing biological species, $\Omega\subseteq\mathbb{R}^n\left(n\geq 1\right)$ is a bounded smooth region, represents the areas where both species live. $a_i,b_i,c_i,d_i\left(i=1,2\right)$ are all positive constants, and d_1,d_2 denote the stochastic dispersal rates of u and v respectively, a_1,a_2 denote the intrinsic growth rates of species u and v respectively, b_1,c_2 denote the number of intra-population competitions for species u and v respectively, c_1,b_2 denote the coefficients of competition between the two populations of species u and v respectively, ρ_{11},ρ_{22} denote the rate of self-expansion within the populations of

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two species respectively. ρ_{12}, ρ_{21} denote the cross-dispersal rates of species u and v, which represent the reproductive pressure between the two species. The habitat Ω is a bounded smooth region in $\mathbb{R}^n$, and is the unit outward normal vector to its smooth boundary $\partial\Omega$.

When the cases without self-diffusion and cross-diffusion terms are not taken into account, $\rho_{11} = \rho_{22} = \rho_{12} = \rho_{21} = 0$, and considering Neumann boundary conditions, homogeneous Neumann boundary conditions represent no exchange between species and the external environment at the boundary of the region Ω, then the system 1 degenerates into the classical Lotka-Volterra competition model.

$$\begin{cases} u_t = d_1 \Delta u + u\left(a_1 - b_1 u - c_1 v\right), & x \in \Omega, t > 0 \\ v_t = d_2 \Delta v + v\left(a_2 - b_2 u - c_2 v\right), & x \in \Omega, t > 0 \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0 \end{cases} \tag{2}$$

Where ν is the out-of-unit normal vector.

The model 2 was initially independently proposed by American population biologist Lotka in 1921 in his study of chemical reaction systems and by Italian mathematician Volterra in 1923 when considering competition among marine fish species. After the 1970s, this model gained widespread attention. Kan-on proved in[3] that the Lotka-Volterra system has no non-constant number of positive steady state solutions under Neumann boundary conditions, and every non-negative solution tends to a constant steady-state solution. However, this conclusion does not hold under "strong competition" conditions. Furthermore, the paper[4] studied the existence of steady-state solutions when the region is convex and proved that under "strong competition" conditions, there are no stable non-constant positive steady-state solutions. Sharp criteria for diffusion and extinction were also obtained. More research results on the Lotka-Volterra competition model can be found in[2,15,17] and their references.

It is worth mentioning that this model is not only significant in ecological diversity but also finds numerous applications in population control, population dynamics, infectious diseases, chemistry, and other fields. In order to further explore the dynamics of the model after incorporating spatial factors, this paper investigates the dynamic characteristics of the Lotka-Volterra system with and without diffusion terms at zero and nonzero solutions based on dynamic transition theory. The aim is to obtain the PES conditions for transitions occurring in the system, as well as the expressions of the new solutions and the types of the jumps. Unlike most of the previous research methods, the main technique in this paper is the dynamical transition theory proposed by Ma and Wang[9,12].The core idea of this theory is to identify all transition (bifurcation) states, which can further analyze the spatial dynamics between populations, thus providing a theoretical foundation for research on biodiversity control.

It is worth mentioning that steady-state bifurcation theory and dynamic transition theory have been widely applied to many intriguing mathematical and physical problems[1,6,7,8,10,11]. To enrich the scope of our work and enhance its practical value, we further investigated the threshold conditions for transitions occurring in the system. In addition to theoretical analysis, some numerical simulations have also played a crucial role in supporting our theory[5,13,16]. Therefore, to enhance the rigor of the article, we also conducted numerical simulations for the system under nonzero solution conditions.

The structure of the article is as follows. Section 3 explores the dynamic transition of the Lotka-Volterra model with Laplacian terms, calculating the PES conditions for transitions occurring at zero and nonzero solutions, determining the thresholds for local stability of the system at non-negative equilibrium points, and demonstrating the occurrence of continuous transitions when the bifurcation parameter exceeds the threshold. Section 4 is dedicated to illustrating the aforementioned theory through numerical simulations.

2. The dynamic transition of the system with two diffusion terms

When the system contains two Laplacian terms, i.e., $\rho_{ij} = 0$ for $i, j = 1, 2$, the system 1 can be transformed into the following equation system

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$$\begin{cases} u_t = \Delta d_1 u + u\left(a_1 - b_1 u - c_1 v\right), & x \in \Omega, t > 0, x \in \left(0, L\right), \\ v_t = \Delta d_2 v + v\left(a_2 - b_2 u - c_2 v\right), & x \in \Omega, t > 0, x \in \left(0, L\right), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0 \end{cases} \tag{3}$$

We investigate the equation (3) under the following initial and boundary value conditions.

$$\begin{aligned} u\left(x,0\right) &= u^0 - u_0, v\left(x,0\right) = v^0 - v_0, \\ u\big|_{\partial\Omega} &= v\big|_{\partial\Omega} = 0. \end{aligned} \tag{4}$$

2.1 The case at the zero solution of the system

Mathematical manipulation: Let $\omega = (u, v)^T$, the equation can be transformed into

$$\frac{d\omega}{dt} = L_\lambda \omega + G\omega$$

Where the linear part is given by

$$L_\lambda = A_\lambda + B_\lambda$$

$$A_\lambda = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix}, B_\lambda \begin{pmatrix} a_1 I & 0 \\ 0 & a_2 I \end{pmatrix}$$

The nonlinear part is given by

$$G\omega = \begin{pmatrix} -b_1 u^2 - c_1 uv \\ -b_2 uv - c_2 v^2 \end{pmatrix}$$

The sectorial operator in the spectral theory of its linear completely continuous field is

$$\begin{cases} -\Delta \psi_k = \rho_k \psi_k, \\ \dfrac{\partial \psi_k}{\partial n}\bigg|_{\partial n} = 0. \end{cases} \quad \left(\psi_k = \cos\frac{k\pi x}{L}\right)$$

Discretize the Laplace term into a sequence

$$\left\{-\rho_k\right\} = -\frac{k^2\pi^2}{L}\left(k = 1, 2, 3 \cdots\right),$$

Then the Jacobi matrix of the system is

$$M_k = \begin{pmatrix} a_1 - d_1\rho_k & 0 \\ 0 & a_2 - d_1\rho_k \end{pmatrix} \tag{5}$$

Noting that the eigenvalue of the Jacobi matrix A of system is $\beta_i\left(i = 1, 2\right)$, its eigenvalue is next calculated by the eigenequation.

$$\left|\beta I - M_k\right| = \begin{vmatrix} \beta - a_1 + d_1\rho_k & 0 \\ 0 & \beta - a_2 + d_2\rho_k \end{vmatrix}$$

The eigenvalues are calculated as

$$\beta_1 = a_1 - d_1\rho_k, \beta_2 = a_2 - d_2\rho_k \tag{6}$$

Since β_1 and β_2 are both parameter-dependent and can change signs, there isn't a definite negative eigenvalue. Therefore we conclude that the zero solution is unstable and the PES condition is not satisfied in this case.

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2.2 The case at the non-zero solution of the system

Let

$$\begin{cases} u' = u - u^* \\ v' = v - v^* \end{cases}$$

Then

$$\begin{cases} u = u' + u^* \\ v = v' + v^* \end{cases}$$

Substituting it into the system (3), for simplicity, omit the derivative symbols, and we can obtain

$$\begin{cases} u_t = d_1 \Delta u - b_1 u^* v - c_1 u^* v - b_1 u^2 - c_1 uv - b_1 u^{*2} + a_1 u^* - c_1 u^* v^*, x \in \Omega, t > 0 \\ v_t^{'} = d_2 \Delta v - b_2 v^* u - c_2 v^* v - c_2 v^2 + a_2 v^* - b_2 u^* v^* - b_2 uv - c_2 v^{*2}, x \in \Omega, t > 0 \end{cases} \tag{7}$$

Mathematical treatment: Let $\omega = (u, v)^T$, the equation can be transformed into

$$\frac{d\omega}{dt} = L_\lambda \omega + G\omega$$

Where the linear part is represented as

$$L_\lambda = A_\lambda + B_\lambda$$

Where

$$A_\lambda = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix}, B_\lambda = \begin{pmatrix} -b_1 u^* & -c_1 u^* \\ -b_2 v^* & -c_2 v^* \end{pmatrix}$$

The nonlinear part is

$$G\omega = \begin{pmatrix} -b_1 u^2 - c_1 uv \\ -c_2 v - b_2 uv \end{pmatrix}$$

Using the sectorial operator from the spectral theory of linear completely continuous fields

$$\begin{cases} -\Delta \psi_k = \rho_k \psi_k, \\ \left. \dfrac{\partial \psi_k}{\partial n} \right|_{\partial n} = 0 \end{cases} \quad \left(\psi_k = \cos \frac{k\pi x}{L} \right)$$

Discretizing the Laplace term into a sequence

$$\{-\rho_k\} = -\frac{k^2 \pi^2}{L}, k = 1, 2, 3 \ldots$$

Then the Jacobian matrix of the system is given by

$$M_k = \begin{pmatrix} -d_1 \rho_k - b_1 u^* & -c_1 u^* \\ -b_2 v^* & -c_2 v^* - d_2 \rho_k \end{pmatrix} \tag{8}$$

The eigenvalues of the Jacobian matrix A for system (3) are denoted as $\beta_i \left(i = 1, 2 \right)$. Then we calculate these eigenvalues using the characteristic equation.

$$\left| M_k - \beta I \right| = \begin{vmatrix} -d_1 \rho_k - b_1 u^* - \beta & -c_1 u^* \\ -b_2 v^* & -c_2 v^* - d_1 \rho_k - \beta \end{vmatrix}$$

The characteristic equation is given by

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$$\beta^2+\left(d_1\rho_k+d_2\rho k+b_1u^*+c_2v^*\right)\beta+\left[\left(d_1\rho_k+b_1u^*\right)\left(c_2v^*+d_2\rho_k\right)-b_2c_1u^*v^*\right]=0$$

The discriminant of the characteristic equation is given by

$$\begin{aligned}\Delta &=\left(d_1\rho_k+d_2\rho_k+b_1u^*+c_2v^*\right)^2-4\left[\left(d_1\rho_k+b_1u^*\right)\left(d_2\rho_k+c_2v^*\right)-b_2c_1u^*v^*\right]\\ &=\left(d_1\rho_k+d_2\rho_k\right)^2+\left(b_1u^*+c_2v^*\right)^2+2\left(d_1\rho_k+d_2\rho_k\right)\left(b_1u^*+c_2v^*\right)\\ &\quad-4\left[\left(d_1\rho_k+b_1u^*\right)\left(d_2\rho_k+c_2v^*\right)-b_2c_1u^*v^*\right]\\ &=\left(d_1\rho_k-d_2\rho_k+b_1u^*-c_2v^*\right)^2+4b_2c_1u^*v^*\end{aligned}$$

The equation has two roots

$$\begin{aligned}\beta_{k_1}&=\frac{-\left(d_1\rho_k+d_2\rho_k+b_1u^*+c_2v^*\right)+\sqrt{\Delta}}{2}\\ \beta_{k_2}&=\frac{-\left(d_1\rho_k+d_2\rho_k+b_1u^*+c_2v^*\right)-\sqrt{\Delta}}{2}<0\end{aligned}\tag{9}$$

The linear characteristic equation of system (3) is given by

$$\begin{aligned}d_1\Delta u-b_1u^*u-c_1u^*v&=\beta u\\ d_2\Delta v-b_2v^*u-c_2v^*v&=\beta v.\end{aligned}\tag{10}$$

Then, we have

$$M_k(\lambda)\eta_{ki}=\beta_{ki}\eta_{ki}$$

and thus the eigenvectors e_{ki} corresponding to $\beta_{ki}(i=1,2)$ from the characteristic equation (3.2.4) can be expressed as

$$e_{ki}=\eta_{ki}\psi_k\tag{11}$$

Its eigenvectors are

$$\begin{aligned}e_{k1}&=\left(\psi_k,\frac{c_2v^*+d_2\rho_k-d_1\rho_k-b_1u^*-\sqrt{\Delta}}{2c_1u^*}\psi_k\right)\\ e_{k2}&=\left(\psi_k,\frac{c_2v^*+d_2\rho_k-d_1\rho_k-b_1u^*+\sqrt{\Delta}}{2c_1u^*}\psi_k\right)\end{aligned}\tag{12}$$

Since

$$\left|M_k^T-\beta I\right|=\begin{vmatrix}d_1\rho_k+b_1u^*+\beta & b_2v^*\\ c_1u^* & c_2v^*+\beta+d_1\rho_k\end{vmatrix}$$

Then

$$e_{k1}^*=\left(\frac{\left(c_2v^*+d_2\rho_k-d_1\rho_k-b_1u^*\right)+\sqrt{\Delta}}{2\sqrt{\Delta}}\psi_k,-\frac{c_1u^*}{\sqrt{\Delta}\psi_k}\right)$$

$$e_{k2}=\left(-\frac{\left(c_2v^*+d_2\rho_k-d_1\rho_k-b_1u^*\right)-\sqrt{\Delta}}{2\sqrt{\Delta}},\frac{c_1u^*}{\sqrt{\Delta}\psi_k}\right).$$

The eigenvectors e_{ki} and their conjugate vectors e_{ki}^* satisfy the relation

$$\left\langle e_i,e_i^*\right\rangle=\begin{cases}1, & i=j\\ 0, & i\neq j.\end{cases}\left(i,j=1,2\right).$$

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Now let's discuss the signs of the two eigenvalues.

Since $\beta_{k1} < 0$, let's now discuss the sign of β_{k2}. Let

$$c_1 = \lambda, \lambda_c = \frac{\left(d_1\rho_k + b_1u^*\right)\left(d_2\rho_k + c_2v^*\right)}{b_2u^*v*},$$

Then the PES condition for system (5) is given by

$$\beta_{k1}\left(\lambda\right) < 0, k = 2,3,4\cdots$$

$$\beta_{11}\begin{cases} <0, & 0<\lambda<\lambda_c \\ =0, & \lambda=\lambda_c \\ >0, & \lambda>\lambda_c. \end{cases} \tag{13}$$

$$\beta_{k2} < 0, k = 1,2,3,4\cdots$$

2.3 Main results

Theorem 2 If the PES condition (13) holds, then the system (3) satisfies the following conclusions:

(1) When $\lambda < \lambda_c$, the equilibrium state $\omega = (0,0)$ of the system is asymptotically stable.

(2) When $\lambda > \lambda_c$, the equilibrium state $\omega = (0,0)$ of the system loses stability.

(3) When $\lambda > \lambda_c$, the system bifurcates into two singular points at $\omega = (0,0)$:

$$x_{11}^{\pm} = \pm\sqrt{\frac{-\beta_{k1}}{\alpha(\lambda)}} + o\left(\sqrt{\beta_{k1}}\right)$$

Proof: We divided the proof into the following three steps.

Step 1: Spatial decomposition

The spaces H and H_1 can be decomposed as

$$H = E_1 \oplus E_2, H_1 = E_1 \oplus \overline{E_2},$$

Where

$$E_1 = span\{e_{11}\}, E_2 = span\{e_{k1}(k \neq 1), e_{k2}\}.$$

Near $\lambda = \lambda_c$, the solution of equation can be expressed as

$$\omega = \omega_1 + \omega_2 \tag{14}$$

Where

$$\omega_1 = x_{11}e_{11} \in E_1, \omega_2 = \sum_{k\neq 1}^{\infty} x_{k1}e_{k1} + \sum_{k\neq 1}^{\infty} x_{k2}e_{k2} \in E_2$$

Furthermore, in the space E_1, equation can be approximated as

$$\begin{aligned} \left\langle e_{11}, e_{11}^* \frac{d_{x_{11}}}{dt} \right\rangle &= \left\langle L_\lambda(\omega), e_{11}^* \right\rangle + \left\langle G(\omega), e_{11}^* \right\rangle \\ &= \beta_{11}\left\langle e_{11}, e_{11}^* \right\rangle x_{11} + \left\langle G(\omega), e_{11}^* \right\rangle \end{aligned} \tag{15}$$

Step 2: Center flow approximation

Through calculation,

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$$\int_0^L (\psi_{11})^3 \, dx = \int_0^L \left(\cos\frac{\pi x}{L}\right)^3 dx = 0$$

So under this condition, we need to consider the influence of the central manifold reduction function

$$\Phi\left(x_{11}e_{11}\right): E_1 \to E_2$$

which means, in (15), we get

$$\omega = x_{11}e_{11} + \Phi\left(x_{11}e_{11}\right)$$

Upon calculation, we obtain

$$\begin{aligned} x_{k1} &= x_{11}^2 \int_0^L \frac{(\cos x)^2}{L^3} \cos kx dx P_1 = Q_1 x_{11}^2, \\ x_{k2} &= x_{11}^2 \int_0^L \frac{(\cos x)^2}{L^3} \cos kx dx P_2 = Q_2 x_{11}^2. \end{aligned} \tag{16}$$

In the vicinity of λ_c, an approximate expression for the central manifold function can be obtained as follows

$$\begin{aligned} \Phi\left(x_{11}e_{11}\right) &= -\sum_{k\neq 1}^{\infty} \frac{x_{k1}}{\beta_{k1}} e_{k1} + \sum_{k\neq 1}^{\infty} \frac{x_{k2}}{\beta_{k2}} e_{k2} \\ &= \begin{pmatrix} -\sum_{k\neq 1}^{\infty} \frac{Q_1}{\beta_{k1}} \psi_k - \sum_{k\neq 1}^{\infty} \frac{Q_2}{\beta_{k2}} \psi_k \\ -\sum_{k\neq 1}^{\infty} \frac{Q_1}{\beta_{k1}} \frac{A-\sqrt{\Delta}}{2c_1 u^*} \psi_k - \sum_{k\neq 1}^{\infty} \frac{Q_2}{\beta_{k2}} \frac{A+\sqrt{\Delta}}{2c_1 u^*} \psi_k \end{pmatrix} x_{11}^2 \end{aligned} \tag{17}$$

Then the central manifold approximation function can be expressed simplistically as $\Phi\left(x_{11}e_{11}\right) = \left(\phi_1, \phi_2\right)^T x_{11}^2$.

Let $B = c_2 v^* - \left(d_1 + d_2\right)\rho_k - b_1 u^*$, then calculate $G\left(x_{11}e_{11}\right)$

$$x_{11}e_{11} = \left(1, \frac{B-\sqrt{\Delta}}{2c_1 u^*}\right) x_{11}\psi_k,$$

$$G\omega = \begin{pmatrix} -b_1 u^2 - c_1 uv \\ -c_2 v^2 - b_2 uv \end{pmatrix}.$$

It can be concluded that

$$G\left(x_{11}e_{11}\right) = \begin{pmatrix} -b_1\left(x_{11}\psi_1\right)^2 - \frac{B-\sqrt{\Delta}}{2u^*}\left(x_{11}\psi_1\right)^2 \\ \frac{-c_2(B\sqrt{\Delta})^2}{4c_1^2 u^{*2}}\left(x_{11}\psi_1\right)^2 - \frac{-b_2\left(B-\sqrt{\Delta}\right)}{2c_1 u^*}\left(x_{11}\psi_1\right)^2 \end{pmatrix}$$

Given conditions that

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$$\begin{aligned}\left\langle G\left(x_{11}e_{11}\right),e_{k1}^{*}\right\rangle &= \left(x_{11}\psi_1\right)^2\psi_k\left(\frac{b_2\left(B-\sqrt{\Delta}\right)-b_1\left(B+\sqrt{\Delta}\right)}{2\sqrt{\Delta}}+\right.\\ &\left.\frac{c_2(B-\sqrt{\Delta})^2-c_1\left(B-\sqrt{\Delta}\right)\left(B+\sqrt{\Delta}\right)}{4c_1u^{*}\sqrt{\Delta}}\right)=P_1\\ \left\langle G\left(x_{11}e_{11}\right),e_{k2}^{*}\right\rangle &= \left(x_{11}\psi_1\right)^2\psi_k\left(\frac{b_1\left(B-\sqrt{\Delta}\right)-b_2\left(B-\sqrt{\Delta}\right)}{2\sqrt{\Delta}}\right.\\ &\left.+\frac{c_1(B-\sqrt{\Delta})^2-c_2(B-\sqrt{\Delta})^2}{4c_1u^{*}\sqrt{\Delta}}\right)=P_2\end{aligned} \tag{18}$$

After calculation, the last term in the reduced equation 15 is

$$\frac{\left\langle G\left(x_{11}e_{11}+\Phi\left(x_{11}\right)\right),e_{11}^{*}\right\rangle}{\left\langle e_{11},e_{11}^{*}\right\rangle}=\alpha\left(\lambda\right)x_{11}^{3}+o\left(3\right), \tag{19}$$

Then

$$\alpha\left(\lambda\right)=Q_3\psi_1^2, Q_3=P_3\phi_1-P_4\phi_2$$

$$P_3=\left(\frac{b_2\left(B-\sqrt{\Delta}\right)-2b_1\left(B+\sqrt{\Delta}\right)+2c_2\left(B-\sqrt{\Delta}\right)}{2\sqrt{\Delta}}+\frac{c_1\left(B-\sqrt{\Delta}\right)\left(B+\sqrt{\Delta}\right)}{4c_1u^{*}\sqrt{\Delta}}\right)$$

$$P_4=\left(\frac{2b_2c_1u^{*}-c_1\left(B+\sqrt{\Delta}\right)}{2\sqrt{\Delta}}\right)$$

Joining 19 and 15, the following approximate equations are obtained.

$$\begin{aligned}\frac{d_{x_{11}}}{dt} &= \beta_{11}x_{11}+\frac{\left\langle G\left(x_{11}e_{11}+\Phi\left(x_{11}\right)\right),e_{11}^{*}\right\rangle}{\left\langle e_{11},e_{11}^{*}\right\rangle},\\ &= \beta_{11}x_{11}+\alpha\left(\lambda\right)x_{11}^{3}+o\left(3\right).\end{aligned} \tag{20}$$

Step 3: Divergence analysis

By solving the equations, we can obtain that when $\lambda > \lambda_c$, the system bifurcates into two singular points at $\omega=\left(0,0\right)$, and the solutions bifurcated are

$$x_{11}^{\pm}=\pm\sqrt{\frac{-\beta_{k1}}{\alpha\left(\lambda\right)}}+o\left(\sqrt{\beta_{k1}}\right).$$

Given that the bifurcation and its local topological structure of equation 7 are determined by the reduced equation (3.3.7), we have

$$\omega_{11}^{\pm}=\pm\sqrt{\frac{-\beta_{k1}}{\alpha\left(\lambda\right)}}e_{11}+\phi\left(\pm\sqrt{\frac{-\beta_{k1}}{\alpha\left(\lambda\right)}}\right)^2+o\left(\beta_{k1}\right),$$

It is a bifurcation solution of equation 7, and the stability of the bifurcation solution $\omega_{11}^{\pm}$ is the same as that of $x_{11}^{\pm}$.

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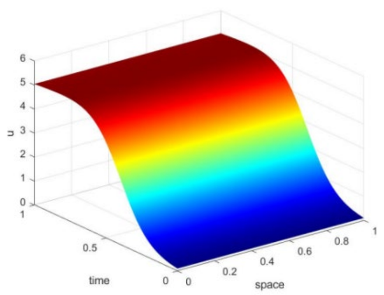
3. Numerical simulation

In this section, we primarily employ numerical simulation methods to conduct a straightforward simulation of the system, aiming to intuitively visualise the type of phase transition and the structure of the system. Previously, when discussing the model, we divided it into cases with and without diffusion terms for discussion. Within the discussions of both cases, we further examined situations at zero and nonzero solutions. Next, we will also conduct a simple simulation of the system's behavior at nonzero solutions without the influence of diffusion terms and the phase diagram of a system with two Laplacian terms.

First, we choose $d_1 = 1$, $d_2 = 1$, $a_1 = 1$, $a_2 = 1$, $b_1 = 0.1$, $b_2 = 0.07$, $c_1 = 0.08$, $c_2 = 0.1$, $u(0,x) = v(0,x) = 0.1$, $\lambda = 0.07$, $\lambda_c = 0.4432697$. In this case, $\lambda < \lambda_c$. We simulate the situation of the system with two Laplacian terms at nonzero solutions, and the results are as follows.



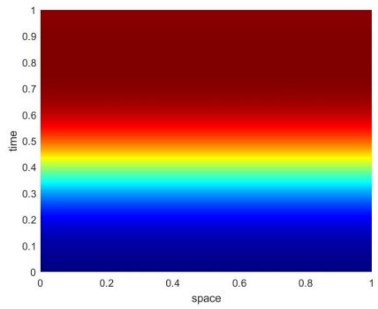
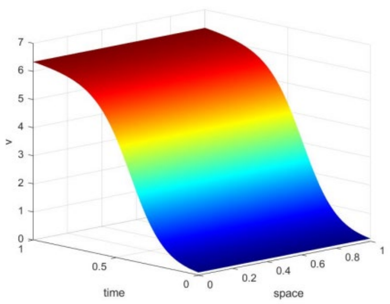


Figure 1: Figure of population u when the control parameter is set to 0.07 (left)

Figure 2: Top view corresponding to Figure 3 (right)



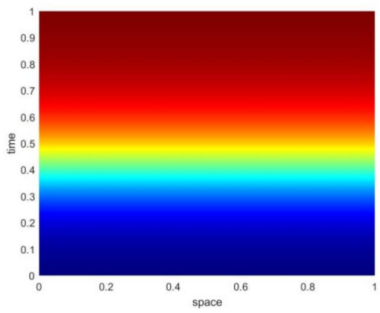


Figure 3: Figure of population v when the control parameter is set to 0.07 (left)

Figure 4: Top view corresponding to Figure 5 (right)

Figure 1 shows the changes in the spatial distribution of population u, and Figure 2 shows the changes of population u at different locations over time, Figure 3 shows the changes in the spatial distribution of population v, and Figure 4 shows the changes of population v at different locations over time. As discussed in the previous calculation, when the system is at a nonzero solution $D(u \cdot v)$ and $0 < \lambda < \lambda_c$, the system is stable. Now, after simulating the phase diagram, both populations stably coexist in the graph, satisfying the PES conditions, consistent with the conclusion drawn in our discussion.

Next, we choose $d_1 = 1$, $d_2 = 1$, $a_1 = 15$, $a_2 = 10$, $b_1 = 0.1$, $b_2 = 9$, $c_1 = 10$, $c_2 = 0.2$, $u(0,x) = v(0,x) = 0.1$, $\lambda_c = 6.9549$. In this case, $\lambda > \lambda_c$. We simulate the situation of the system with two Laplacian terms at nonzero solutions.

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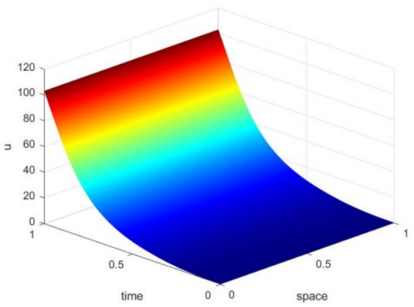


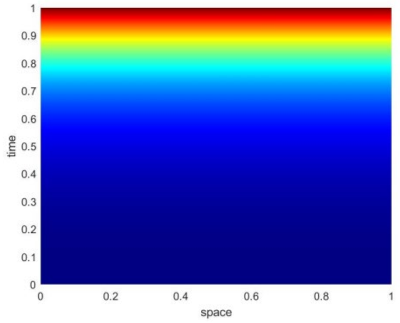
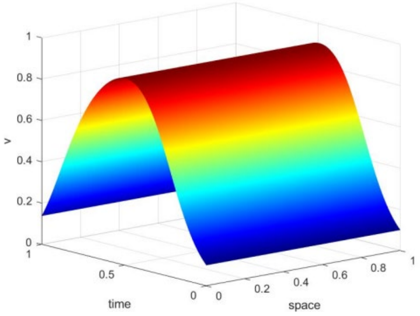


Figure 5: Figure of population u *when the control parameter is set to 9 (left)*

Figure 6: Top view corresponding to Figure 7 (right)



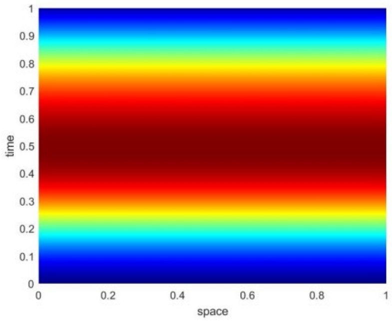


Figure 7: Figure of population v *when the control parameter is set to 9 (left)*

Figure 8: Top view corresponding to Figure 9 (right)

Figure 5 shows the changes in the spatial distribution of population u, and Figure 6 shows the changes of population u at different locations over time, Figure 7 shows the changes in the spatial distribution of population v, and Figure 8 shows the changes of population v at different locations over time. As discussed in the previous calculation, when the system is at a nonzero solution $D(u'v)$ and $0\langle\lambda\rangle\lambda_c$, the system loses stability. Now, after simulating the phase diagram, it is observed that the PES conditions are not satisfied. The two populations do not coexist at non-negative solutions, indicating a transition, consistent with the conclusion drawn in our discussion.

From the figures above, when the system is at nonzero solution $D(u^*,v^*)$, and $\lambda > \lambda_c$、$u(0,x) = v(0,x) = 2\sin(x\pi)$, system stability is the same as the situation when $u(0,x) = v(0,x) = 0$, failing to satisfy the PES Conditions. The two populations do not coexist at nonnegative solutions; a transition occurs, consistent with the conclusion reached in the course of our discussions. All the numerical simulations above confirm the conclusions drawn in our paper.

4. Conclusion

Based on the theory of dynamic transition and considering spatial factors, this paper conducts a dynamic analysis of the Lotka-Volterra model with and without diffusion terms. We obtain the PES conditions and stability thresholds at both zero and nonzero solutions of the system. Furthermore, we demonstrate that the system undergoes continuous transitions when the bifurcation parameter exceeds the stability threshold. Finally, we validate our theory through numerical simulations using the finite difference method. The results of this study can be applied in fields such as biodiversity. In future work, we plan to conduct further analysis, such as considering the impact of factors like time delay.

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