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Solving String Vibration Problems Based on Partial Differential Equations

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Abstract: *This article focuses on using the method of separating variables and Fourier transform to solve string vibration problems. Starting from the research significance of string vibration problems and the study of partial differential equations, the advantages, disadvantages, and applicability of the method of separating variables and Fourier transform are compared. The method of separating variables and Fourier transform are used to solve one-dimensional wave, two-dimensional wave equation, and three-dimensional wave equation problems, providing a way of thinking for studying multidimensional wave equations.*

Keywords: *string vibration; Wave equation; partial differential equation*

1. Introduction

String vibration is a classical problem in physics, the study of string vibration involves the establishment and solution of fluctuation equations, the use of partial differential equations to construct the fluctuation equations, and its research results have promoted the development of mathematical and physical methods. In general, partial differential equations are used to solve the practical problems of string vibration, and the method of separated variables and Fourier transform are generally used. The development of partial differential equations can not only further improve the mathematical theory system, but also promote the partial differential equations to play a greater role in solving practical problems.[1]

With the help and guidance of the teachers, we consolidated the math courses in high school, studied Calculus, Functions of Complex Variables, Analytic Geometry, Ordinary Differential Equations, Advanced Algebra, Integral Transforms, and part of Recent Algebra, Probability Theory, Mathematical Statistics, and then, after communicating with Prof. Peng Wu, and combining with our personal interests, we established our research direction as the study of string vibration problems by partial differential equations. [2]

In this paper, starting from the research significance of string vibration problems and the research methods of partial differential equations, we explore the use of the separated variable method and the Fourier transform method to solve one-dimensional fluctuation, two-dimensional fluctuation equations and three-dimensional fluctuation equations.

2. Research methodology

Table 1 Comparison of the characteristics of two methods

Characteristics	The split-variable method	Fourier transform method
Pros	The physical significance is obvious	Wide applicability
	Precise analytical solutions	simplifying the solution process
	Simple calculations	dealing with complex fluctuation equations
Disadvantages	limited scope of application	physical significance is not apparent
	The solution is a hierarchy of	High computational costs
	dependent boundary conditions	

Separation of variables method and Fourier transform method are two common methods to solve the string vibration problems Table 1. By assuming that the solution is in the form of a product of different variables, the separated variables method transforms the one-dimensional fluctuation equation into

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several ordinary differential equations, and then utilizes the method of solving ordinary differential equations to solve the fluctuation equations one by one. The method of separating variables generally has the following characteristics.[3]

Pros

■ Physical significance is obvious: Separate variable methods are directly related to the geometry and boundary conditions of the problem, and the solutions obtained have obvious physical significance.

■ Exact analytical solutions: Exact analytical solutions can be obtained under certain conditions, such as simple geometries and chi-square boundary conditions.

■ Simple calculation: The calculation process is relatively simple compared to the Fourier transform method, which has a standardized solution.

Disadvantages

■ Limited scope of adaptation: In general, it is only applicable to problems with chi-square boundary conditions, and other additional processing is required if the boundary conditions are not chi-square.

■ The solution is a cascade: The solution of the separated variables method is often a cascade, complicating the expression of the solution in cases where numerical computation is required.

■ Dependent Boundary Conditions: Dependent on the form of boundary conditions, for non-standard boundary conditions, special treatment is required.

The Fourier transform method has the following characteristics.

Pros

■ Wide applicability: applicable to a variety of initial conditions and boundary conditions of the problem, can effectively deal with complex boundary problems.

■ Simplify the solution process: Simplify the solution process by converting a partial differential equation problem into an ordinary differential equation problem.

■ Handling complex fluctuation equations: Fluctuation equations with nonlinear terms can be solved efficiently by transforming them into algebraic operations in the frequency domain.

Disadvantages

■ Physical significance is not obvious: the solution obtained by the Fourier transform method is generally an expression in the frequency domain, and the physical significance is not obvious.

■ High computational cost: A large number of integrals and steps are involved, resulting in high computational cost.

3. The main study

String vibration problems are usually based on the assumptions that the string is uniform and soft, that its vibration is small and occurs in a plane, and that the displacements of points on the string are perpendicular to the equilibrium position. Under these assumptions, Newton's second law and Hooke's law can be used to establish a mathematical model of string vibration.[4]

The string vibration fluctuation equation can be expressed as

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Among them. $u(x,t)$ denotes the string at position , x and time t The displacement of the a is the speed of the wave, which is related to the tension and density of the string.

$$a^2 = \frac{T}{\rho} \tag{2}$$

T It's the tension in the string. ρ is the line density of the string.

The problem of fluctuation equations

The main research is to solve the fluctuation equations by using the separated variable method and Fourier transform method. It mainly involves five types of one-dimensional fluctuation equations, two types of two-dimensional fluctuation equations, and three types of three-dimensional fluctuation equations. The two-dimensional fluctuation equations only consider the solutions of bounded equations, and the three-dimensional fluctuation equations only consider the solutions of chi-square equations. In the research process, we feel the advantages and disadvantages of the two methods to solve the string vibration fluctuation equations, and master the two methods to solve the string vibration fluctuation equations[5].

4. Findings and analysis

4.1 One-dimensional fluctuation equations

One-dimensional fluctuation equations: according to the difference of boundary and initial value conditions, the one-dimensional fluctuation equations are divided into five categories, including one-dimensional bounded chi-squared fluctuation equations, one-dimensional bounded damped fluctuation equations, one-dimensional bounded telegraphic equations, one-dimensional bounded non-chiral fluctuation equations, and one-dimensional unbounded non-chiral fluctuation equations.

■ One-dimensional bounded chi-squared fluctuation equations

The fluctuation equation for a one-dimensional bounded chi-squared fluctuation equation is given by

$$\left\{\begin{aligned} & u_{tt} = a^2 u_{xx} \\ & u\big|_{x=0} = 0, u\big|_{x=l} = 0 \\ & u\big|_{t=0} = u(x,0) = \phi(x), \frac{\partial u}{\partial t}\bigg|_{t=0} = \frac{\partial u}{\partial t}(x,0) = \psi(x) \end{aligned}\right. \tag{3}$$

The first step of the solution separates the variables, obtaining

$$\left\{\begin{aligned} & \mathrm{x}'' + \lambda x = 0 \\ & T'' + a^2 \lambda T = 0 \end{aligned}\right. \tag{4}$$

And then we input separately to $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.Each of the three cases is brought to the boundary conditions,the first two cases, the $x = 0$ rounding, third $\lambda < 0$.The case continues to be brought to the initial value condition, and the solution of the equation is obtained.

■ One-dimensional bounded damped fluctuation equations

The fluctuation equations change because of damping, and the one-dimensional bounded-damped fluctuation equations,

$$\text{the}\left\{\begin{aligned} & u_{tt} - a^2 u_{xx} + 2bu_t = 0 \\ & u\big|_{x=0} = 0, u\big|_{x=l} = 0 \\ & u\big|_{t=0} = \phi(x), \frac{\partial u}{\partial t}\bigg|_{t=0} = \psi(x) \end{aligned}\right. \tag{5}$$

The first step of the solution separates the variables, obtaining

$$\left\{\begin{aligned} & \mathrm{x}'' + \lambda x = 0 \\ & T'' + 2bT' + \lambda T = 0 \end{aligned}\right. \tag{6}$$

And then separately to $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ Three cases are discussed, the first two remain rounded, and the third $\lambda < 0$ situation combined with the initial value condition score $0 < \lambda < b^2$,

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$\lambda = b^2$, $\lambda > b^2$ Solve for each of the three cases.

■ One-dimensional bounded telegraph equations

The fluctuation equation of the one-dimensional bounded telegraph equation continues to change,as, the

$$\left\{\begin{array}{l} u_{tt} - a^2 u_{xx} + 2bu_t + cu = 0 \\ u\big|_{x=0} = 0, u\big|_{x=l} = 0 \\ u\big|_{t=0} = \phi(x), \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \end{array}\right. \tag{7}$$

First separating the variables to get that

$$\left\{\begin{array}{l} \mathrm{x}'' + \lambda x = 0 \\ T'' + 2bT' + (c + \lambda a^2)T = 0 \end{array}\right. \tag{8}$$

And then too $\lambda > 0$, $\lambda = 0$ Give it up. $\lambda < 0$ "Order $g = (c + \lambda a)^2 - b^2$ Points $g < 0$, $g = 0$, $g > 0$ Solve separately in conjunction with the initial value condition.

■ One-dimensional bounded nonchiral fluctuation equations

The one-dimensional bounded nonchiral fluctuation equations are such that

$$\left\{\begin{array}{l} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t) \\ u\big|_{x=0} = \partial(t), u\big|_{x=l} = p(t) \\ u\big|_{t=0} = \phi(x), \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \end{array}\right. \tag{9}$$

■ One-dimensional unbounded nonchiral fluctuation equations

The one-dimensional unbounded nonchiral fluctuation equations are such that

$$\left\{\begin{array}{l} u_{tt} - a^2 u_{xx} = f(x,t) \\ u\big|_{t=0} = \phi(x), \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \end{array}\right. x \in R, t > 0 \tag{10}$$

The key to both of these is to separate the variables by taking non-chiral equations and using the principle of chiralization to find non-chiral initial values.

4.2 Two-dimensional fluctuation equations

■ Bounded fluctuation equations in two dimensions: rectangular boundaries

Two-dimensional bounded fluctuation equations:Rectangular boundaries (non-chiral terms for $f(x,y,t)$) The fluctuation equation is given by

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = f(x,y,t) \text{nonhomogeneous equation} \\ u\big|_{x=0} = \partial_1(y,t), u\big|_{x=a} = \partial_2(y,t) \\ u\big|_{y=0} = \beta_1(x,t), u\big|_{y=b} = \beta_2(x,t) \qquad \text{boundary} \\ u\big|_{t=0} = \phi(x,y), \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x,y) \text{Initial value} \end{cases} \tag{11}$$

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The solution is.

The first step is a linear superposition of the equations, which makes the boundary chi-squared.

Step 2, Chiralization of equations (chirality principle, eigenfunction method)

The third step is the initial value problem for the boundary chi-square of the equations.

■ Bounded fluctuation equations in two dimensions: circular boundaries

Two-dimensional bounded fluctuation equation:The circular boundary fluctuation equation is given by

$$\begin{cases} u_{tt} - c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) = f(r,\theta,t) \\ u\big|_{r=l_1} = g_1(\theta)\, u\big|_{r=l_2} = g_2(\theta) \\ u\big|_{t=0} = \phi(r,\theta), \frac{\partial u}{\partial t}\bigg|_{t=0} = \psi(r,\theta) \end{cases} \tag{12}$$

The solution is.

The first step is the boundary chiralization, a superposition of the equations linearly.

Step 2, Chiralization of equations (chirality principle, eigenfunction method)

The third step is the initial value problem for the boundary chi-square of the equations.

4.3 Three-dimensional fluctuation equations

To study the solution of three-dimensional fluctuation equations in the form of rectangular coordinates, only the solution of chi-square three-dimensional fluctuation equations is discussed in order to simplify the complexity. In the following, according to the different boundary conditions, the three-dimensional fluctuation equations are solved in three categories.[6]

■ Three-dimensional chi-square fluctuation equations:x,y,z are unbounded

The three-dimensional chi-square fluctuation equations with no boundary at x,y,z are given by

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), t > 0 \\ u(x,y,z,0) = \phi(x,y,z), -\infty < x,y,z < \infty \\ u_t(x,y,z,0) = \psi(x,y,z) \end{cases} \tag{13}$$

The basic solution is first solved, then superimposed using the superposition theorem, for both sides of the equation variables, the x, y, z Perform a Fourier transform solution.

■ Three-dimensional chi-square fluctuation equations:x,y,z are bounded

The three-dimensional chi-squared fluctuation equations with boundaries at x,y,z are given by

$$\begin{gathered} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}) \\ 0 \le x \le a, 0 \le y \le b, 0 \le z \le c \end{gathered} \tag{14}$$

Firstly, it is necessary to separate the temporal and spatial variables, and then use the boundary conditions to solve the problem by the separated variable method.

■ Three-dimensional chi-squared fluctuation equations:x,y,z are not fully bounded

The three-dimensional chi-squared fluctuation equation for x,y,z that is not fully bounded is given by

$$\begin{gathered} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}) \\ 0 \le x \le a, 0 \le y \le b, -\infty \le z \le \infty \end{gathered} \tag{15}$$

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The study of fluctuation equations has deepened the understanding of the chi-squaredization theorem and the method of separating variables. Under different boundary and initial value conditions, the three-dimensional fluctuation equations are much more difficult to solve than the one- and two-dimensional fluctuation equations, but roughly speaking, they are all trying to transform the higher dimensions into the lower dimensions, and the mathematical thinking of the essence is all the same.

5. The main innovations

Study of the systematic nature of solving the fluctuation equation problem: This paper is based on the one-dimensional to three-dimensional equations, not only solving the fluctuation of a single dimension, but from one-dimensional to three-dimensional, exploring the optimal method of solving the fluctuation equations of the string vibration at the present stage, and laying the foundation for solving the multidimensional string vibration problem and providing some new ideas.

Considering damped fluctuation equations: In the study of one-dimensional fluctuation equations, the damping term is added to the fluctuation equations by considering the fluctuations in damped media.

6. Conclusions

By using the method of separated variables and Fourier transform method, the one-dimensional to three-dimensional fluctuation equations are solved. What needs to be optimized is that, in order to reduce the difficulty of the study, the two-dimensional fluctuation equations only consider the two-dimensional bounded fluctuation equations, and the three-dimensional fluctuation equations only consider the three-dimensional chi-squared fluctuation equations, and in the future study, focus on studying the fluctuation equations of two-dimensional and three-dimensional unbounded fluctuation equations. In the study of two-dimensional fluctuation equations, two-dimensional fluctuation equations are generally solved as follows, the first step of the boundary chi-squared, the equations linear superposition. In the second step, the equations are chiralized. The third step is to solve the initial value of the boundary chirality of the equation. For three-dimensional fluctuation equations: x, y, z are not bounded, the use of the Fourier transform method, the main consideration is that the basic solution can be found using the superposition theorem, so that the Fourier transform can be solved directly, greatly simplifying the solution process.

The following aspects should be kept in mind in solving the problem of fluctuation equations for string vibration.

First of all, in-depth understanding of mathematical concepts and definitions, if combined with the actual physical meaning will be remembered more firmly and for a longer period of time, in-depth understanding can help learners to better apply to the actual. Secondly, to cultivate mathematical intuition, mathematical intuition can help the researcher to quickly determine which method should be used to solve the string vibration fluctuation problem, and play a multiplier effect. Lastly, the researcher should be skilled in consulting related materials, cultivate the ability to use mathematical resources to assist learning, and summarize the learning methods and gains on a regular basis.

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