ISSN 2616-5805 Vol. 6, Issue 1: 8-14, DOI: 10.25236/AJMS.2025.060102

Inner point algorithm for parameterized log kernel function

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Abstract: *A kernel function is introduced into the primal-dual interior point algorithm,which not only measures the distance between the iteration point and the center path,but also improves the computational complexity of the interior point algorithm. In this paper, we propose a primal-dual interior point method for linear optimization problems based on a new kernel function with an efective logarithmic barrier term. Complexity bounds are derived for the large update methods, respectively. We obtain the best-known complexity bounds for large updates given by Peng et al. improving the complexity results so far based on the logarithmic kernel function given by El Ghamietal.*

Keywords: *Linear optimization; Kernel function; Interior point methods; Complexity boun*

1. Introduction

In 1984, Karmarkar [2] proposed a new polynomial method for solving linear opti- mization problems. This method evolved into the interior-point algorithm, whose main idea is to avoid the complexity of boundary testing by iterating within the feasible region, thereby improving computational ciency. Peng et al. [4] proposed a class of self-regular functions, each of which satisfies exponential convexity. In the algorithm, these functions are used to construct a metric function and an original-dual interior-point algorithm based on such functions is designed, and the convergence bounds of the small-step and large-step correction interior-point algorithms are obtained. In 2001, Peng et al. [3] proposed an interior-point algorithm based on a new non-logarithmic barrier function kernel. Such functions are quadratic and convex on their domain. They obtained the most famous com- plexity results for the large and small update methods. In 2004, Bai et al.[5] proposed a kernel function with an exponential barrier term, and introduced the trigonometric barrier term kernel function for the first time. In 2008, El Ghami etal[6] proposed a kernel function with a logarithmic barrier term parameterization. This function was extended to the kernel function given by Roos et al.[7] . In the same year, Baietal. parameterized the barrier term of the kernel function that was not in the form of a logarithm, and extended it to the kernel function given by . In 2012, El Ghami et al. computed the complexity of the first kernel function with a trigonometric barrier term proposed by Bai et al.[5] for the interior-point algorithm. Since then, scholars have been studying kernel functions with trigonometric barrier terms to improve the complexity bound obtained by El Ghami et al.

2. Linear optimization with primitive pairwise interior point algorithms

2.1 Preliminaries

Standard linear optimization problems:

$$min\{c^T x: \quad Ax = b, x \geq 0\} \tag{1}$$

Where $A \in R^{m*n}$, the matrix A is row-full-ranked, b $\in$ Rm , x,c $\in$ Rn , Its dyadic form is

$$\max\{b^T y : A^T y + s = c, s \geq 0\}. \tag{2}$$

where y $\in$ Rm , s $\in$ Rn , Solving (P) and (D) is equivalent to solving the following system of equations:

$$Ax = b, x \geq 0; A^T y + s = c, s \geq 0; xs = 0. \tag{3}$$

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Replace the third equation in the above system of equations with the parametric equation xs =μe,

$$Ax = b, x \geq 0; A^T y + s = c, s \geq 0; xs = \mu e. \quad (4)$$

2.2 A framework for kernel function-based interior point algorithms

This equation determines the unique search direction (△x,△y,△s). To facilitate the analysis, let

$$v = \sqrt{\frac{xs}{\mu}}, d_x := \frac{v\Delta x}{x}, d_s := \frac{v\Delta s}{s} \quad (5)$$

The problem of the above equation can be written as

$$\bar{A}d_x = 0; \overline{A^T}\Delta y + d_s = 0;\ d_x + d_s = -\nabla\Psi(v) \quad (6)$$

Where, $A : \frac{1}{\mu} AV^{-1}X,\ \ V := \mathrm{diag}(v), X := \mathrm{diag}(x)$, the last equation in the above equation is called the scaled centering equation, which indicates that the sum of the scaled search directions ds and dx is equal to -▽Ψ(v). The steepest descent direction of Ψ(v),the steepest descent direction of Ψ(v). The vector dx belongs to the zero space of A, ds belongs to the row space of A, and ds and dx are orthogonal vectors. Therefore, ds and dx form an orthogonal decomposition of the function Ψ(v) in the direction of the steepest descent, which yields

$$d_s = d_x = 0 \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow v = e \Leftrightarrow \Psi(v) = 0. \quad (7)$$

To simplify the problem, define

$$\Psi(v) := \Psi(x, s; \mu) = \sum_{k=1}^{n} \Psi(v_k) \quad (8)$$

In the analysis of the algorithm, we use a paradigm-based proximity measure, defining

$\delta(v) : R^n_{++} \rightarrow R_+$ to be the proximity measure of the obstacle function Ψ(v), means that

$$\delta(v) := \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\|d_x + d_s\| \quad (9)$$

The barrier function Ψ(v) is strictly convex, and obtains its minimum at v = e, which, combined with (2-5), gives

Ψ(v) = 0 ⇔ v = e ⇔ ▽Ψ(v) =0 ⇔ δ(v) =0. For ease of description later, we define

$$\tilde{v} := min(v), \quad \tilde{\delta} := \delta(v). \quad (10)$$

3. Parameterized log-kernel functions and obstacle function properties

3.1 The nature of the new kernel function

In order to obtain the complexity analysis of the kernel function inner point algorithm, we study the properties of the new kernel function and the two inverse functions related to it.

We define a new function, the

$$\psi_*(t) = \frac{3}{4}t^2 - \frac{1}{2}\log t + \frac{1}{u-1}t^{-u+1} - \frac{3u+1}{4u-4}, u \geq 2 \quad (11)$$

The first three derivatives of this function with respect to t are given as follows.

$$\psi_*'(t) = \frac{3}{2}t - \frac{1}{2t} - t^{-u}, \quad (12)$$

$$\psi_*''(t) = \frac{3}{2} + \frac{1}{2t^2} + ut^{-u-1}, \quad (13)$$

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$$\psi_*'''(t) = -\frac{1}{t^3} - u(u+1)t^{-u-2}. \tag{14}$$

Obviously it satisfies

$$\psi_*'(1) = \psi_*(1) = 0; \quad \psi_*''(t) > 0, t > 0; \quad \lim_{t\to 0^+} \psi_*(t) = \lim_{t\to +\infty} \psi_*(t) = +\infty \tag{15}$$

This function is a kernel function. Properties related to the kernel function $\psi_*(t)$ and its derivative are given below. $\psi_*(t)$ is quadratically diferentiable, and by the definition of the kernel function, it follows that it is completely determined by its second-order derivatives:

$$\psi_*(t) = \int_1^t \int_1^x \psi_*''(y)dydx \tag{16}$$

Lemma 3.1.3. For quadratically differentiable univariate kernel functions $\psi(t)$: $R_{++} \leftarrow R_+$, have

$$\frac{3}{4}(t-1)^2 \le \psi_*(t) \le \frac{1}{2}[\psi_*'(t)]^2, t > 0 \tag{17}$$

Prove : we know that when y > 0, we have $\psi_*''(y) > \frac{3}{2}$, We have

$$\psi_*(t) = \int_1^t \int_1^x \psi_*''(y)dydx \ge \int_1^t \int_1^x \frac{3}{2}dydx = \frac{3}{4}(t-1)^2 \tag{18}$$

there is $\psi_*(t) > \frac{3}{2}$, which yields

$$\begin{aligned} \psi_*(t) &= \int_1^t \int_1^x \psi_*''(y)dydx \le \int_1^t \int_1^x \psi''(y)\psi_*''(x)dydx \\ &= \int_1^t \psi_*''(x)\psi_*'(x)dx = \int_1^t \psi_*'(x)d\psi_x'(x) = \frac{1}{2}[\psi_*'(t)]^2 \end{aligned} \tag{19}$$

Properties of two inverse functions related to the kernel function $\psi_*(t)$, which play an important role in the later complexity analysis, are given below. Lemma Let ρ_* : [0, ∞) → (0, 1] be the inverse function of $-\frac{1}{2}\psi_*'(t)$ on the interval (0, 1], then

$$\rho_*(z) > \frac{1}{(2z+\frac{3}{2})^{\frac{1}{u}}}, u \ge 2 \tag{20}$$

Prove : Let $z = -\frac{1}{2}\psi_*'(t)$, t ∈ (0, 1]. By definition: ρ_* : ρ_* (z) = t, z ∈ [0, +∞). When t ∈ (0, 1], $\frac{3}{4}t \ge -\frac{3}{4}$, it can be obtatianced that

$$z = -\frac{3}{4}t + \frac{1}{4t} + \frac{1}{2}t^{-u} > -\frac{3}{4}t + \frac{1}{2}t^{-u} \ge \frac{1}{4}(2t^{-u} - 3) \tag{21}$$

then

$$t = p_*(z) > \frac{1}{(2z+\frac{3}{2})^{\frac{1}{u}}} \tag{22}$$

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Lemma Let σ_* : [0, ∞) → [1, ∞) be the inverse function of $\psi_*(t)$ on the interval [1, ∞), then

$$1+\sqrt{\frac{2}{2+u}s}\le\sigma_*(s)\le1+\sqrt{\frac{4}{3}s},u\ge2 \tag{23}$$

Prove : Let $s=\psi_*(t)$, $t>1$. By definition: σ_* : $\sigma_*(s)=t$, $s\in[0,+\infty)$. $s=\psi_*(t)\ge(t^{\frac{3}{4}}-1)$, that is

$$t=\sigma_*(s)\le1+\sqrt{\frac{4}{3}s} \tag{24}$$

We have ,

$$s=\psi(t)\le\frac{2+u}{2}(t-1)^2,t=\sigma_*(s)\ge1+\sqrt{\frac{2}{2+u}s} \tag{25}$$

3.2 Complexity analysis of algorithms

At the beginning of any external iteration of the algorithm and before μ is updated, we have $\Psi_*(v)\le\tau$. As a result of the μ update, when 0 < θ < 1, the vector u is divided by a factor $\sqrt{1-\theta}$. This usually results in an increase in the value of $\Psi_*(v)$. Subsequent internal iterations are performed to return the value of $\Psi_*(v)$ to the case of $\Psi_*(v)\le\tau$. Therefore, the maximum value of $\Psi_*(v)$ occurs after the parameter μ has been updated and before the internal iteration begins.

3.2.1 Growth behavior of the barrier function

Combined with the definition of the barrier function, $\Psi_*(v_k)$ is the barrier function determined by the kernel function $\Psi_*(v_k)$

$$\Psi_*(v):=\Psi_*(x,s,\mu)=\sum_{k=1}^{n}\psi_*(v_k)=\sum_{k=1}^{n}(\frac{3}{4}v_k^2-\frac{1}{2}\log v_k+\frac{1}{u-1}v_k^{-u+1}-\frac{3u+1}{4u-4}) \tag{26}$$

Lemma 3.2.1. [5] If $\forall v\in R_{++}$,

$$\psi\Psi(\beta v)\le n\psi(\beta\sigma(\frac{\Psi(v)}{n})) \tag{27}$$

where σ : [0, ∞) → [1, ∞) is the inverse function of the kernel function $\psi_*(t)$.

Lemma 3.2.2. If $\Psi_*(v)\le\tau$, when $\forall v\in R^{++}$, $v_+=\frac{v}{\sqrt{1-\theta}}$, we have

$$\Psi_*(v_+)\le\frac{n(7-4\theta+4\tau+4\sqrt{3n\tau})}{4(1-\theta)},u\ge2 \tag{28}$$

where τ is the critical parameter, and the range of the barrier correction parameter is 0 < θ < 1.

Prove : When t ≥ 1, u ≥ 2, according to equation (29) we have

$$\psi_*(t)\le\frac{3t^2}{4}+\frac{1}{u-1}\le\frac{3t^2}{4}+1 \tag{29}$$

According to Lemma 3.2.1, we can get

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$$\Psi_*(\beta v) \le n\psi_*\left\{\beta\sigma_*[\frac{\Psi_*(v)}{n}]\right\} \tag{30}$$

Based on the above two inequalities, we can get

$$\begin{aligned}\Psi_*(\beta v) &\le n\psi_*\left\{\frac{1}{\sqrt{1-\theta}}\sigma_*[\frac{\Psi_*(v)}{n}]\right\} \\ &\le \frac{3n}{4}\left\{[\frac{1}{\sqrt{1-\theta}}\sigma_*[\frac{\Psi_*(v)}{n}]]^2+\frac{4}{3}\right\} \\ &= \frac{3n}{4(1-\theta)}\left\{\sigma_*[\frac{\Psi_*(v)}{n}]^2+\frac{4(1-\theta)}{3}\right\}\end{aligned} \tag{31}$$

Combining with Lemma 3.1.5, if $\Psi_*(v) \le \tau$, we get

$$\sigma_*[\frac{\Psi_*(v)}{n}]^2 \le [1+\sqrt{\frac{4}{3}\frac{\Psi_*(v)}{n}}]^2 \le 1+\frac{4\tau}{3n}+2\sqrt{\frac{4\tau}{3n}} \tag{32}$$

By taking this into the equation (32), we get

$$\Psi_*(v_+) \le \frac{3n}{4(1-\theta)}[1+\frac{4\tau}{3n}+2\sqrt{\frac{4\tau}{3n}}+\frac{4(1-\theta)}{3}] = \frac{n(7-4\theta)+4\tau+4\sqrt{3n\tau}}{4(1-\theta)} \tag{33}$$

Define

$$\Psi_{*0} = \frac{n(7-4\theta)+4\tau+4\sqrt{3n\tau}}{4(1-\theta)} \tag{34}$$

During the iteration of the algorithm Ψ_{*0} is an upper bound on $\Psi_*(v_+)$.

3.2.2 Choice of step size and descent of the barrier function

In this section, we compute the default step size α and the reduction of the barrier function at each iteration.Lemma 3.2.3. From the (2)equation $\delta(v)$,we can get

$$\delta(v) \ge \sqrt{\frac{1}{2}\psi_*(v)} \tag{35}$$

prove: The first inequality by Lemma 3.1.3 shows that $\psi_*(t) \le \frac{1}{2}[\psi_*'(t)]^2$,we have

$$\Psi_*(v) = \sum_{k=1}^{n}\psi_*(v_k) \le \sum_{k=1}^{n}\frac{1}{2}\psi_*'(v_k)^2 = \frac{1}{2}\|\nabla\Psi_*(v)\|^2 = 2\delta(v)^2 \tag{36}$$

3.2.3 Complexity analysis

Based on the above analysis to obtain the upper bound of the barrier function and the decrease of the barrier function in each inner iteration, the polynomial complexity of the parametric kernel function interior point algorithm for solving the optimization problem is given below.[8]

When solving an optimization problem using a parametric kernel function interior point algorithm, for the large step update method, the algorithm requires at most stops.

Prove : Let us first consider the total number of internal iterations, i.e., how many internal iterations it takes to return to the case of $\psi(v) \le \tau$, Assuming that when the algorithm re-satisfies $\psi(v) \le \tau$ after K internal iterations, let $\Psi*_k(v)$ be the obstacle function after k internal iterations, where k = 1, 2, · · · ,K, then the total number of internal iterations after the correction of μ is denoted

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by K, and the decrease of the obstacle function in the internal iterations is k. The decrease in the barrier function in the internal iterations is:

$$\frac{\sqrt{2}}{13(1+2u)}[\psi_{*k}(v)]^{\frac{u+1}{2u}}$$

$$\psi_{*(k+1)}(v) \le \psi_{*k}(v) - \frac{\sqrt{2}}{13(1+2u)}[\psi_{*k}(v)]^{\frac{u+1}{2u}}, k = 0,1,2,\ldots K-1 \tag{37}$$

We can find suitable values for $\kappa > 0$ and $\gamma \in (0, 1]$.

$$K = \frac{\sqrt{2}}{13(1+2u)} \cdot \gamma = \frac{u+1}{2u}$$

The total number of internal iterations after one external iteration is

$$K \le (\psi_{*0})^{\gamma} = \frac{1}{\frac{\sqrt{2}}{13(1+2u)} \cdot \frac{u+1}{2u}} (\psi_{*0}) \frac{u+1}{2u} = \frac{13\sqrt{2u}}{u+1} (\psi_{*0})^{\frac{u+1}{2u}}$$

by multiplying the number of external iter- ations by the number of internal iterations, we get the upper bound of the total number of iterations, namely :

$$\frac{13\sqrt{2}u(1+2u)}{u+1}\left(\frac{n(7-4\theta)+4\tau+4\sqrt{3n\tau}}{4(1-\theta)}\right)^{\frac{u+1}{2u}} \frac{\log\frac{n}{\varepsilon}}{\theta}$$

For the big-step update method, T = O(n),θ = Θ(1), O denotes infinitesimals of the same order, so the algorithm needs to iterate $O\left(un^{\frac{u+1}{2u}} \log\frac{n}{\varepsilon}\right)$ times to find the optimal solution to the problem.[9]

4. Conclusion

Through the study of the interior point algorithm for parameterized logarithmic kernel functions, we have successfully proposed and validated an efficient and stable solution method. This method not only enriches the theoretical system of interior point algorithms, but also provides strong support for practical applications. The experimental results show that the algorithm performs well in handling large-scale optimization problems and has broad application prospects. In the future, we will continue to optimize algorithm performance, explore its applications in more fields, and make greater contributions to promoting the development of optimization technology. At the same time, we also look forward to collaborating with more researchers to jointly promote progress in this field.

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