Academic Journal of Computing & Information Science
ISSN 2616-5775 Vol. 5, Issue 3: 17-22, DOI: 10.25236/AJCIS.2022.050302

New Solution of Equation of Degree n with One Unknown

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Abstract: *There are many methods to solve equation of degree n with one unknown, but the restrictions of most methods are very harsh. Based on Vieta's theorem, this paper gives the symbolic effective coefficient method and the elimination of rotation symmetry method to solve the n-th equation of one variable.*

Keywords: *Symmetry of functional displacement, Vieta's theorem, Equation of degree n with one unknown, factorization of polynomials*

1. Introduction

As early as 1830, Galois proved that there is no algebraic solution to the equation of degree 5 and above in general [1-2], and the research on this problem has been stagnant for a long time. Until modern times, some scholars have studied the algebraic solution of univariate higher order equation. The numerical solution of the higher order equation can be obtained by using programming software [3], but the traditional factorization method and transformation method [4] are still used in the algebraic solution. this kind of method has strict requirements for the coefficient of the equation and is not suitable for the solution of general cases.

Based on the shortcomings of the above methods, the unary n-th equations are divided into three categories according to the types of roots, which can be transformed into the case that the equations are all single root after classification and discussion. In this paper, two methods for solving the univariate n-th equation are given: one is the symbolic effective coefficient method of finding the root step by step, and the root can be found item by item only by giving the special sum of the coefficients. The other is a theoretically feasible method to eliminate rotation symmetry [5-6]. By adding conditions to eliminate the rotation symmetry of the system of equations formed by Weida's theorem, in order to achieve the purpose of finding its unique solution. Therefore, this paper focuses on the transformation of the equation with multiple roots, solving the univariate n-degree equation with single root and how to add conditions to eliminate the rotation symmetry of the equation.

2. Necessary specification of algebraic equations

Lemma 1: Any unary n_{th} degree equation $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0(a_0\neq 0)$ can be translated so that the coefficient of n-1 degree term is 0.

$$f(x-\frac{a_1}{na_0})=a_0x^n+C_n^1a_0x^{n-1}(-\frac{a_1}{na_0})+a_1x^{n-1}+g(x)=a_0x^n+g(x) \tag{1}$$

$g(x)$ is the $x^k(k\leq n-2)$ linear combination of $f(x-\frac{a_1}{na_0})$, that is $\partial(g(x))<n-1$ or g(x)=0

Definition 1: If only all the roots of *f*(*x*)=0 are composed of all the single roots of the univariate equation *g*(*x*)=0 of degree n, then *g*(*x*)=0 is called the essential equation of *f*(*x*)=0. Specially, if $\left(f(x),f^{'}(x)\right)=f^{'}(x)(\partial(f(x))\geq 2)$, then the essential factor of f(x) is g(x)=1, There is no essential equation. If $\left(f(x),f^{'}(x)\right)=1$, then the essential factor of f(x) is f(x)=0

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Definition 2: Any $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0(a_0\neq 0)$, name the a_i/a_0 effective coefficient, and the $(-1)^i\frac{a_i}{a_0}(i=1,2,\ldots,n)$ is the symbolic efficiency coefficient. Obviously, if the equation has even degree, the total symbolic efficiency coefficient of f(x)=0 is $\sum_{i=1}^{n}(-1)^i\frac{a_i}{a_0}=\frac{f(-1)}{a_0}-1$. If the degree of the equation is odd, the total symbolic efficiency coefficient of f(x)=0 is $\sum_{i=1}^{n}(-1)^i\frac{a_i}{a_0}=-\frac{f(-1)}{a_0}-1$.

3. Symmetry of rotation

Rotation symmetry is not only widely used in multiple integrals, but also has important research value in solving the uniqueness of equations and equations. Now we need to simply define rotation symmetry.

Definition 3: From solvable equation with n unknowns $f(x_1,\ldots x_i,\ldots x_j,\ldots x_n)=0$, if the relative positions of the two unknowns are swapped $f(x_1,\ldots x_i,\ldots x_j,\ldots x_n)=0$ and $f(x_1,\ldots x_j,\ldots x_i,\ldots x_n)=0$ have the same expression. Thus name x_i and x_j have rotation symmetry with respect to equation f.

It is clear from the definition that the following conclusion can be reached:

Lemma 2: If xi and xj have rotation symmetry with respect to f, under the condition of $xi=xj$, the equation $f(x_1,\ldots x_i,\ldots x_j,\ldots x_n)=0$ can be unique.

Lemma 3: If any two unknowns in the system of equations have rotation symmetry with respect to the system of equations and the unknowns are not completely equal, then the system of equations is solvable and the solution is not unique.

4. A arrangement in the field of complex numbers

The following is a way of ordering complex numbers in which all complex numbers (including imaginary numbers with non-zero imaginary parts) can be given positions and all complex numbers can be sorted. Sort as follows:

$\forall\alpha,\ \beta\in C,\ and\ \ \alpha=a_1+b_1i,\beta=a_2+b_2i$, that is:

(1) If $a_1>a_2, then\ \alpha\succ\beta$

(2) If $a_1=a_2,b_1<b_2,\ then\ \alpha\prec\beta$

(3) If $a_1=a_2,b_1=b_2,\ then\ \alpha\approx\beta$

In fact, this arrangement is the priority relationship between the real part and the imaginary part of a given complex number, first compare the size of the real part, and then compare the size of the imaginary part if the real part is equal. In particular, when the imaginary parts of two complex numbers are zero, this permutation relation is actually Archimedean property over the real number field.

5. General ideas for solving unary NTH degree equations

It is easy to solve the equation of one degree, this paper only studies the equation of two degree or more. According to the type of roots, the unary n-degree equation is divided into three cases to be discussed below:

Arbitrary unary nth power equation $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0(a_0\neq 0)$, and $(f(x),f^{'}(x))=d_1(x)$

(1) If $d_1(x)=1$, then f(x)=0 are all single roots.

(2) If $\forall x_0 \in \{x \mid f(x)=0\}, that\ f^{'}(x_0)=0$, then all the roots of f(x)=0 are double roots.

(3) If $\exists x_0 \in \{x \mid f(x)=0\}, that\ f^{'}(x_0) \neq 0 \cap d(x) \neq 1$, then the root of f(x)=0 has both single root and double root.

5.1. The root of f(x) is all single

According to lemma 1, we may as well assume that the first term coefficient is 1, n-1 power of the general unary n_{th} degree equation coefficient is 0:

$$f(x)=x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0(a_1=0) \tag{2}$$

Weida's theorem states the relationship between the roots of the equation and the coefficients, equation f(x)=0 roots x_i(i=1,2,…,n)

$$\begin{cases} f_1(x_1,x_2,\ldots x_n)=x_1+x_2+\ldots+x_n=(-1)^1a_1=0 \\ f_2(x_1,x_2,\ldots x_n)=x_1x_2+x_1x_3+\cdots+x_1x_n+\cdots+x_{n-1}x_n=(-1)^2a_2 \\ \cdots \\ f_n(x_1,x_2,\ldots x_n)=x_1x_2\ldots x_{n-1}x_n=(-1)^na_n \end{cases} \tag{3}$$

5.1.1. Symbolic efficiency coefficient method

According to the corollary of the remainder theorem, we can get $f(x)=(x-x_1)f_1(x)$, suppose:

$$\begin{cases} x_2+x_3+\ldots+x_n=U_1 \\ x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n=U_2 \\ \cdots \\ x_2\ldots x_{n-1}x_n=U_{n-1} \end{cases} \tag{4}$$

The original system of equations is obtained by the substitution of variables:

$$\begin{cases} -U_1^2+U_2=(-1)^2a_2 \\ -U_1U_2+U_3=(-1)^3a_3 \\ \cdots \\ -U_1U_{n-2}+U_{n-1}=(-1)^{n-1}a_{n-1} \\ -U_1U_{n-1}=(-1)^na_n \end{cases} \tag{5}$$

Obviously, $\sum_{i=2}^{n}(-1)^i a_i=\sum_{i=1}^{n}(-1)^i a_i(a_1=0)$ is the symbolic effective coefficient sum. $\sum_{j=1}^{n-1}U_j$ is the symbolic effective coefficient sum of $f_1(x)=0$. Since we need to describe the sum of symbolic efficient coefficients later, we only discuss the case of $f(-1)\neq 0$; if f(-1)=0, using the corollary of the first-order remainder theorem, the discussion of $g(x)=\frac{f(x)}{x+1}(g(-1)\neq 0)$ is consistent with the case of $f(-1)\neq 0$.

If n is odd, f_1(x)=0 has an even degree: $\sum_{i=1}^{n}(-1)^i a_i=-f(-1)-1, \sum_{j=1}^{n-1}U_j=f_1(-1)-1$.

Because $f(-1)=-(1+x_1)f_1(-1)$, $\sum_{i=1}^{n}(-1)^i a_i=(1+x_1)(1+\sum_{j=1}^{n-1}U_j)-1$.

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$$x_1 = \frac{\sum_{i=1}^{n}(-1)^i a_i - \sum_{j=1}^{n-1} U_j}{1+\sum_{j=1}^{n-1} U_j} \tag{6}$$

If n is even, f1(x)=0 has an odd degree, the calculated x1 is consistent with the answer that n is odd, so the value has nothing to do with the parity of n. In order to facilitate the representation of xi(i=1,2,…,n), then set $A_0 = \sum_{i=1}^{n}(-1)^i a_i, A_1 = \sum_{j=1}^{n-1} U_j$, then $x_1 = \frac{A_0 - A_1}{1+A_1}$.

In the same way $f_1(x) = (x - x_2) f_2(x)$, we get $x_2 = \frac{A_1 - A_2}{1+A_2}$ after the same operation. Generally, $x_r = \frac{A_{r-1} - A_r}{1+A_r} (r = 1,2,\ldots,n-1)$, $x_n = A_{n-1}$.

(Note: $f_{r-1}(x) = (x - x_r) f_r(x)$, the symbolic effective coefficient sum of A_r is f_r(x))

The advantage of symbolic effective coefficient method and traditional factorization method is that we do not need to know all the coefficients of the equation, but only need to know their symbolic effective coefficients and can find the root item by item, which can greatly reduce the amount of operation and is very convenient.

5.1.2. Elimination of rotation symmetry

Here is another way to solve the unary n-degree equation, suppose *f*(x)=0 the roots are all simple roots. If you look at the system of equations, you will find that there is rotation symmetry between n unknowns:

$$\begin{aligned} &\forall k \in N^+ \cap [1,n], x_i.x_j \in \{x \mid f(x)=0\}, \exists f_k(x_1 \ldots x_i \ldots x_j \ldots x_n) = 0, \\ &then\ f_k(x_1 \ldots x_i \ldots x_j \ldots x_n) = f_k(x_1 \ldots x_j \ldots x_i \ldots x_n). \end{aligned} \tag{7}$$

According to lemma 3, the solution of the equations is not unique.In order to eliminate the rotation symmetry of the equations with any unknown quantity, the solution of the equations is unique. Because the roots of equation *f*(*x*)=0 are all single, the n roots are not equal to each other in the complex field. We sort the x_i(i=1,2,…,n) according to the sort above. Suppose $x_1 \prec x_2 \prec \ldots \prec x_n$, where the relative positions between the two of the n unknowns of the equations can not be exchanged, that is, the rotation symmetry between all elements is eliminated and the solution of the equations is unique.

Both the method of symbolic effective coefficient and the method of eliminating rotation symmetry have generality compared with the traditional method of solving higher order equations, and there is no special requirement on the coefficient of the expression of f(x)=0.

5.2. The root of f(x) is all single

If all the roots of f(x)=0 are heavy roots, and the d1(x)=f'(x). We might as well do the toss and turn division again: $(d_1(x), d_1'(x)) = d_2(x)$, compare $\partial(d_1(x))$ and $\partial(d_2(x))$ .The number of times the equation is found is reduced, that is, the number of repeated roots is $m = \partial(d_2(x)) - \partial(d_1(x))$. And then we assume that if we do s of toss and turn division, the degree of this equation will go down to 0, and we get $(d_{s-1}(x), d'_{s-1}(x)) = d_s(x) = 1 (s \geq 2)$.

Instead, we find the essential factor of di(x) every time and find the essential factor $g_i(x)(1 \leq i \leq s)$. Obviously, the root of $G(x) = g_1(x) g_2(x) \ldots g_s(x) = 0$ is the root f(x)=0, and $(G(x), G'(x)) = 1$ .Then the solution of the equation containing only multiple roots can be transformed into the solution of the equation containing only single roots G(x)=0 problem, and the solution G(x)=0 can be completely used in the above method.

Any k multiple root of f(x) = 0 will not appear as a solution in the greatest common factor up to the

ISSN 2616-5775 Vol. 5, Issue 3: 17-22, DOI: 10.25236/AJCIS.2022.050302

k at most after the alternation of $d_{i-1}(x)$ and $d_{i-1}'(x)$. Then the multiplicity k=i of any k multiple root x_k of f(x) = 0, and $d_{i-1}(x_k)=0, d_i(x_k)\neq 0$. We correspond the multiplicity of the equation to the multiple roots, then 0) (the case that all the roots of the equation f(x) = 0 are multiple roots is solved by doing the division method of s times. The advantage of this method is that the order of the higher order equation can be reduced without knowing the root of the equation, which greatly reduces the amount of calculation of solving the univariate nth order equation f(x) = 0.

5.3. There are both simple roots and multiple roots of f(x)=0

In general, the equation of higher degree of one variable has both single and multiple roots. This is the most complex case, and we can convert this problem to the first two cases by using the previous method for single and multiple roots.

Suppose p(x)=0 is the essential equation of f(x)=0,

Then the roots of $q(x)=0$ in $f(x)=p(x)q(x)$ are all multiple roots of f(x)=0, and the roots of q(x)=0 are all multiple roots. The equation q(x)=0 is treated according to the case, and the equation G(x) = 0 is obtained after several times of division. If we make F(x)=p(x)G(x)=0, we will find (F(x),F'(x))=1, which means that F(x)=0 is all single root, and we will deal with the equation F(x)=0 on a case-by-case basis.

6. Example

$$f(x)=x^{11}+3x^{10}-13x^9-35x^8+29x^7+111x^6+5x^5-177x^4-50x^3+146x^2+28x-48=0$$

After 4 times of tossing and turning, the following results were obtained:

$$d_1(x)=(f(x), f^{'}(x))=x^4-2x^3+2x-1, d_2(x)=(f^{'}(x), f^{''}(x))=x^2-2x+1$$

$$d_3(x)=(f^{''}(x), f^{'''}(x))=x-1, d_4(x)=(f^{'''}(x), f^{(4)}(x))=1$$

Let $g_i(x)$ be the essential factor of $d_i(x)$, which can be obtained from the definition of essential factor:

$$g_4(x)=1, g_3(x)=\frac{d_3(x)}{d_4(x)}=x-1, g_2(x)=\frac{d_2(x)}{d_3^{\ 2}(x)}=1, g_1(x)=\frac{d_1(x)}{d_2^{\ 3/2}(x)}=x+1$$

Obviously-1 and 1 are the double and quadruple roots of $f(x)$=0.

$$p(x)=\frac{f(x)}{f^{'}(x)G(x)}=x^5+5x^4-2x^3-38x^2-68x-48$$

$$h^*(x)=p(x)G(x)=\frac{f(x)}{f^{'}(x)}=x^7+5x^6-3x^5-43x^4-66x^3-10x^2+68x+48$$

In order to facilitate the discussion of the following equation, the following processing is done:

$$h(x)=\frac{h^*(x)}{x+1}=x^6+4x^5-7x^4-36x^3-30x^2+20x+48(h(-1)\neq 0)$$

At this point, if you want to use the factorization method to decompose the factor $h(x)=(x-x_1)h_1(x)(\partial(h_1(x))=5)$, then you need to know the specific values of the coefficients of h_1(x), we need to know six coefficients. If we use the symbolic effective coefficient method, we only need to know the symbolic effective coefficient of $h_1(x)$=0 and a value of A1, such as: $h_1(x)=x^5+5x^4-2x^3-38x^2-68x-48$, $x-x_1=\frac{h(x)}{h_1(x)}=x-1$.

ISSN 2616-5775 Vol. 5, Issue 3: 17-22, DOI: 10.25236/AJCIS.2022.050302

If we know the symbolic effective coefficient and $A_0 = \sum_{i=1}^{6} (-1)^i \frac{a_i}{a_0} = 23$ of h(x)=0, we need to know the symbolic effective coefficient sum $A_1 = \sum_{j=1}^{5} (-1)^j \frac{b_j}{b_0} = 11$ of $h_1(x) = 0$ (x), then: $x_1 = \frac{A_0 - A_1}{1 + A_1} = 1$

Solve the equation term by term using factorization h(x)=0, we need to know at least 21 numbers, use the term by term sign effective coefficient method, at most only need 6 sign effective coefficient and can achieve the purpose of term by term root.

7. Conclusion

This paper explores how to transform the equation with multiple roots into the equation without multiple roots, to solve the univariate n-th equation with only a single root, and to add conditions to eliminate the rotation symmetry of the equation. Based on Weida's theorem, the symbolic effective coefficient method and the method of eliminating rotation symmetry for solving univariate n-th equation are given.

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