ISSN 2616-5805 Vol. 6, Issue 1: 82-90, DOI: 10.25236/AJMS.2025.060110

Existence Theorems for Linear Mappings on Hypercomplex Number Systems

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Abstract: *This paper delves into the study of vector spaces defined over the ring P, a commutative ring with zero divisors generated by two real numbers. By systematically constructing vector spaces over P, a commutative ring with zero divisors, this research delves deeper into the algebraic framework to comprehensively analyze linear transformations and mappings defined within this context. The study not only establishes foundational principles for the existence and behavior of such mappings but also examines their structural properties and interactions, thereby providing a rigorous framework for further theoretical exploration and application in algebraic systems involving P. Leveraging the unique factorization property of zero divisors in P, the paper establishes an existence theorem for linear mappings on P, thereby providing a significant advancement in understanding the interplay between vector spaces and the structural properties of P. The findings not only deepen theoretical understanding of algebraic structures associated with P but also pave the way for further investigations into the broader implications of such structures. This work contributes a foundational perspective for future research in abstract algebra and its potential applications.*

Keywords: *Vector Space, Linear Transformation, Linear Mapping, perplex numbers*

1. Introduction

Hayes [1] studied the dynamics of hyperbolic numbers, highlighting their significant differences from complex numbers, such as the hyperbolic Mandelbrot set forming a filled square and the topology of the filled Julia set contrasting with the complex case. Blankers [2] explored the Julia and Mandelbrot sets in complex dynamical systems and examined their analogs in hyperbolic number systems, demonstrating how the hyperbolic Mandelbrot set characterizes the connectedness of hyperbolic Julia sets. Kumar [3] introduced various types of double complex sequences, showing that regular convergent sequences are bounded and form Banach spaces, with examples supporting algebraic and topological properties. Atmai and Rachid [4] presented a pedagogical resource that guides undergraduates to explore new algebraic and geometric structures through simple sign changes, fostering creativity and mathematical thinking. The studies examine the unique dynamics of hyperbolic numbers, including their contrasts with complex numbers in dynamical systems. They highlight the algebraic and topological properties of hyperbolic and double-complex structures, emphasizing their educational value and potential for fostering mathematical creativity.

Debrouwere [5] investigated the extension problem for continuous linear maps within (LB)-spaces, establishing conditions for surjectivity and applying the results to tensorized maps and vector-valued Eidelheit-type problems. Kushnir [6] provided conditions for a linear transformation to preserve the intersection of two convex closed sets and extended these results to non-convex sets and continuous transformations, illustrating their practical applications. Liu Yu Jia [7] explored the geometric interpretation of fundamental linear algebra concepts in low-dimensional cases, aimed at helping beginners and laying a foundation for higher-dimensional studies. Fang Longfei [8]examined linear mappings in finite-dimensional spaces, focusing on how the properties of linear transformations and similar matrices can deepen students' understanding through the interplay of abstract and visual thinking. These studies address the extension of linear maps, the properties of intersections in convex sets, and the practical applications of such mappings. They also emphasize the geometric interpretation of fundamental concepts in linear algebra, highlighting the integration of algebraic and geometric thinking in education and research.

Pogorui, Anatoliy A.[9]studied a random motion on a hyperbola with random absolute velocity and two opposite directions of motion, governed by a Poisson switching process, and calculated the probability density function (pdf) of the particle's position. Robson[10] reviewed the fundamental principles of Quantum Mechanics, Quantum Computing, and Artificial Intelligence, linking them to hyperbolic imaginary numbers and their matrices, rediscovered by Dirac, and their applications in quantum computing, quantum field theory, and probability theory. Kobayashi [11] explored the inclusion of hyperbolic numbers in Clifford algebra and the challenges of analyzing hyperbolic neural networks (NNs), which have been extended from Clifford algebra. This study investigates the irreducibilities of hyperbolic NNs and presents a new form of reducibility, hyperbola-reducibility. Tellez Sanchez, G. Y. [12] extended the Chaos game algorithm and Shannon entropy to the hyperbolic number plane, associating these concepts with hyperbolic-valued probabilities. These studies explore the applications of hyperbolic numbers in random motion, quantum computing, neural network analysis, and probability theory, highlighting their broad potential in mathematics, physics, and computational fields. The research also innovatively extends classical algorithms and theories to the domain of hyperbolic numbers.

2. The preliminaries of perplex numbers

Our definition of perplex numbers is as follows:

$$\mathbb{P} = \{\delta = a + bh; a, b \in \mathbb{R}\}, \tag{1}$$

The perplex unit *h* is defined here, and the definition of the perplex unit h is that the perplex unit *h* satisfies $h^2 = 1$ and $h \neq \pm 1$. In past literature, numbers defined in this way are also referred to as double numbers, spacetime numbers, hyperbolic numbers, or split complex numbers. One of the properties of the set $\mathbb{P}$ is that it is a commutative ring with the following operations of addition and multiplication:

$$\begin{aligned}\delta_1 + \delta_2 &= (a_1 + b_1 h) + (a_2 + b_2 h)\\ &= (a_1 + a_2) + (b_1 + b_2)h,\end{aligned} \tag{2}$$

and

$$\begin{aligned}\delta_1 \delta_2 &= (a_1 + b_1 h)(a_2 + b_2 h)\\ &= (a_1 a_2 + b_1 b_2) + (a_1 b_2 + a_2 b_1)h,\end{aligned} \tag{3}$$

Where $(\delta_1 = a_1 + b_1 h)$ and $(\delta_2 = a_1 + b_1 h)$ .For any $\delta = a + bh \in \mathbb{P}$ we define the real part of δ as $\mathrm{Re}(\delta) = a$ and the perplex part of δ as $\mathrm{Im}(\delta) = b$ .The conjugate of δ is denoted by $\bar{\delta}$ and it is $\bar{\delta} = a - bh$. The ring of perplex number has zero--divisors. Also, perplex number can be written as:

$$h^+ = \frac{1+h}{2} \text{ and } h^- = \frac{1-h}{2}, \tag{4}$$

and

$$h^+ h^- = 0. \tag{5}$$

We call $\{h^+, h^-\}$ the idempotent base of $\mathbb{P}$. It is noted that zero divisors are idempotent elements.

$$(h^+)^2 = h^+ \text{ and } (h^-)^2 = h^-, \tag{6}$$

and we have

$$h^+ + h^- = 1, h^+ - h^- = h, \tag{7}$$

They are idempotent zero divisors, that form the whole ring. If $\delta = a + bh \in \mathbb{P}$, then the idempotent representation is

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$$\delta = (a+b)h^{+} + (a-b)h^{-} = uh^{+} + vh^{-}, \tag{8}$$

It is clear that the zero-divisors in the set $\preccurlyeq$ are real multiples of h^{+} and h^{-}.This is because the multiples of h^{+} and h^{-} lie on the lines $y = x$ and $y = -x$. Therefore, δ is a divisor if and only if $\delta = uh^{+}$ or $\delta = vh^{-}, u, v \in \mathbb{R} \setminus \{0\}$. Also, for $\delta_1 = u_1 h^{+} + v_1 h^{-}$ and $\delta_2 = u_2 h^{+} + v_2 h^{-} \in \mathbb{P}$, we have:

$$\delta_1 + \delta_2 = (u_1 + u_2)h^{+} + (v_1 + v_2)h^{-}, \tag{9}$$

$$\delta_1 \delta_2 = (u_1 u_2)h^{+} + (v_1 v_2)h^{-}. \tag{10}$$

The set

$$\mathbb{P}^{+} = \{\delta = uh^{+} + vh^{-}; u \geq 0, v \geq 0\}, \tag{11}$$

is called the set of non-negative perplex numbers. We can see if $\delta_1, \delta_2 \in \mathbb{P}^{+}$,then $\delta_1 \delta_2 \in \mathbb{P}^{+}$.

Therefore, $\mathbb{P}^{+}$ is closed under the multiplication.

So $\delta, \zeta \in \mathbb{P}^{+}$,we have

$$\delta \preccurlyeq \zeta \ \text{ iff } \ \zeta - \delta \in \mathbb{P}^{+}, \tag{12}$$

and we say that ζ is $\mathbb{P}$-greater than δ,or

$$\delta \succcurlyeq \zeta \ \text{ iff } \ \zeta - \delta \in \mathbb{P}^{-}, \tag{13}$$

and we say that ζ is $\mathbb{P}$-less than δ. Also, $\delta \in \mathbb{P}^{+}$ is equivalent to $0 \preceq \delta$ and that $\delta \in \mathbb{P}^{+} \setminus \{0\}$ is equivalent to $0 \prec \delta$. For $\delta_1 = u_1 h^{+} + v_1 h^{-}$ and $\delta_2 = u_2 h^{+} + v_2 h^{-}$,we say that $\delta_1 \preccurlyeq \delta_2$ iff $u_1 \leq u_2$ and $v_1 \leq v_2$.And, the relation $\preccurlyeq$ is reflexive, transitive and antisymmetric ,so, the relation $\preccurlyeq$ denotes a partial order in $\preccurlyeq$. We can see that in Figure 1. this relation order. Here, let the x-axis and y-axis represents a one-dimensional space. These spaces are embedded in $\mathbb{P}$. If the speed of light is taken as 1, the set of zero divisors with a positive real part is called the future direction, and the set of zero divisors with a negative real part is called the past direction. From the points on the

$y = x$ and $y = -x$ lines, the ones with $x > 0$ are the future direction, the ones with $x < 0$ are the past direction . The real line is embedded in $\mathbb{P}$ by injection $\varphi: \mathbb{R} \to \mathbb{P}$, for all $a \in \mathbb{R}$.

$$\varphi(a) = a = ah^{+} + ah^{-}, \tag{14}$$

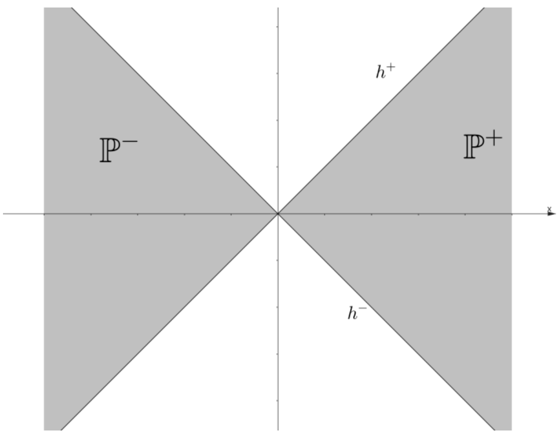

Figure 1. The concept of positive and negative perplex number

ISSN 2616-5805 Vol. 6, Issue 1: 82-90, DOI: 10.25236/AJMS.2025.060110

Definition 1 For $\delta_1 = u_1h^+ + v_1h^-$ and $\delta_2 = u_2h^+ + v_2h^-$ in $\mathbb{P}$ such that $\delta_1 \preccurlyeq \delta_2$, the Closed Perplex interval $[\delta_1, \delta_2]_{\mathbb{P}}$ is defined by

$$[\delta_1, \delta_2]_{\mathbb{P}} = \{\lambda \in \mathbb{P}: \delta_1 \le \lambda \le \delta_2\} \tag{15}$$

Equivalently, $\lambda = uh^+ + vh^- \in [\delta_1, \delta_2]_{\mathbb{P}}$ iff

$$u_1 \le u \le u_2 \text{ and } v_1 \le v \le v_2. \tag{16}$$

We can see the Figure 2. If $\delta_1 - \delta_2$ is a non-negative zero divisor perplex number, then the perplex interval $[\delta_1, \delta_2]_{\mathbb{P}}$ is degenerate. And, if $\delta_1 - \delta_2$ is an invertible positive perplex number, then the perplex interval is non-degenerate. The perplex intervals have the notion of length as in the real interval. The length of any hyperbolic interval is a non-negative perplex number.

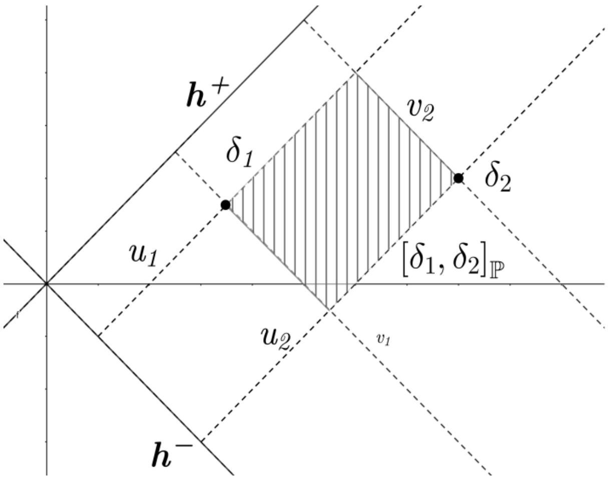

Figure 2. The concept of perplex number intervals.

3. Results

3.1 Vector space in the $\mathbb{P}$-Ring and Their Properties

3.1.1 Proof that Vector Space in the Ring $\mathbb{P}$

The following is proof that the non-empty set V in the ring $\mathbb{P}$ is a vector space.

The proofs of (1), (2), (3), and (4) are analogous to those in advanced algebra.

(5) For $\alpha_1, \alpha_2 \in V$ and all perplex numbers $\beta_1 = c_1h^+ + d_1h^- \in \mathbb{P}$

$$\begin{aligned} \beta_1(\alpha_1 + \alpha_2) &= (c_1h^+ + d_1h^-)(\alpha_1 + \alpha_2) \\ &= (c_1h^+ + d_1h^-)\alpha_1 + (c_1h^+ + d_1h^-)\alpha_2 \\ &= \beta_1\alpha_1 + \beta_1\alpha_2, \end{aligned} \tag{17}$$

(6) For $\alpha_1 \in V$ and for all perplex numbers $\beta_1 = c_1h^+ + d_1h^-$ and $\beta_2 = c_2h^+ + d_2h^- \in \mathbb{P}$

$$\begin{aligned} \alpha_1(\beta_1 + \beta_2) &= \alpha_1(c_1h^+ + d_1h^- + c_2h^+ + d_2h^-) \\ &= \alpha_1(c_1h^+ + d_1h^-) + \alpha_1(c_2h^+ + d_2h^-) \\ &= \alpha_1\beta_1 + \alpha_1\beta_2, \end{aligned} \tag{18}$$

(7) For $\alpha_1 \in V$ and for all perplex numbers $\beta_1 = c_1h^+ + d_1h^-$ and $\beta_2 = c_2h^+ + d_2h^- \in \mathbb{P}$

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$$\begin{aligned}(\beta_1\beta_2)\alpha_1 &= (c_1c_2h^+ + d_1d_2h^-)\alpha \\ &= (c_1h^+ + d_1h^-)(c_2h^+ + d_2h^-)\alpha_1 \\ &= \beta_1[(c_2h^+ + d_2h^-)\alpha_1] \\ &= \beta_1(\beta_2\alpha_1),\end{aligned} \tag{19}$$

(8) For $\alpha_1 \in V$,there exists a perplex number $h^+ + h^- \in \mathbb{P}$

$$(h^+ + h^-)\alpha_1 = \alpha_1, \tag{20}$$

3.1.2 Properties of a Vector Space in the Ring $\mathbb{P}$

Let V be a vector Space in the Ring $\mathbb{P}$. then

(a)The zero element 0 of the linear space V is unique.

(b)For all $\alpha_1 \in V$, exists $0 = 0h^+ + 0h^- \in \mathbb{P}$, such that $0\alpha_1 = 0$

(c) For each $\alpha_1 \in V$, the additive inverse $-\alpha_1$ is unique.

(d) For all $\alpha_1 \in V$, and all $\beta_1 = c_1h^+ + d_1h^- \in \mathbb{P}, (-\alpha_1)\beta_1 = -(\alpha_1\beta_1)$.

Proof:

The proof of (a) is analogous to that in linear algebra.

(b)We have

$$\alpha_1 + 0\alpha_1 = (h^+ + h^- + 0h^+ + 0h^-)\alpha_1 = (h^+ + h^-)\alpha_1 = \alpha_1, \tag{21}$$

Adding $-\alpha_1$ to both sides of the equation yields

$$0\alpha_1 = 0. \tag{22}$$

(c)Assuming α_1 has two negative elements ς and γ then

$$\alpha_1 + \varsigma = 0, \tag{23}$$

$$\alpha_1 + \gamma = 0, \tag{24}$$

Then

$$\varsigma = \varsigma + 0 = \varsigma + (\alpha_1 + \gamma) = (\varsigma + \alpha_1) + \gamma = 0 + \gamma = \gamma. \tag{25}$$

This contradicts the fact that ς and γ are two different negative elements; therefore, the additive inverse $-\alpha_1$ is unique.

(d)We have

$$\beta_1\alpha_1 + (-\beta_1\alpha_1) = [\beta_1 + (-\beta_1)]\alpha_1 = [c_1h^+ + d_1h^- + (-c_1h^+) + (-d_1h^-)]\alpha_1 = (0h^+ + 0h^-)\alpha_1 = 0, \tag{26}$$

by axiom (6) and part(b). Hence $(-\beta_1\alpha_1)$ also serves as an additive inverse for $\beta_1\alpha_1$. By part(c), therefore, we must have $(-\alpha_1)\beta_1 = -(\alpha_1\beta_1)$

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3.2 Linear Mappings in the Ring $\mathbb{P}$ and Their Properties

3.2.1 Definition of a Linear Mappings in the Ring $\mathbb{P}$

Definition 2 A mapping $\mathscr{A}$ from the vector space V to V' is called a linear mapping if, for any elements $\alpha_1, \alpha_2 \in V$, and any scalar $\beta_1 = c_1 h^+ + d_1 h^- \in \mathbb{P}$,the following hold:

$$\mathscr{A}(\alpha_1 + \alpha_2) = \mathscr{A}(\alpha_1) + \mathscr{A}(\alpha_2), \tag{27}$$

$$\mathscr{A}(\beta_1 \alpha_1) = \beta_1 \mathscr{A}(\alpha_1). \tag{28}$$

A linear mapping from vector space *V* to itself is commonly referred to as a linear transformation on *V*

3.2.2 Properties of a Linear Transformation in the Ring $\mathbb{P}$

Let $\mathscr{A}$ be a Linear Transformation in vector Space V then:

(a) $\mathscr{A}(0) = 0,$

(b) For all $\alpha_1 \in V, \mathscr{A}(-\alpha_1) = -\mathscr{A}(\alpha_1)$

(c) For all $\alpha_i \in V, \beta_i = c_i h^+ + d_i h^-, \eta = \sum_{i=1}^{n} \beta_i \alpha_i, \mathscr{A}(\eta) = \sum_{i=1}^{n} \beta_i \mathscr{A}(\alpha_i), (i = 1, 2, 3, \cdots\cdots, n)$

Proof:

(a) From the definition of a linear transformation in the ring $\mathbb{P}$, we have:

$$\mathscr{A}(0) = \mathscr{A}(0 \cdot \alpha_1) = 0 \mathscr{A}(\alpha_1) = 0. \tag{29}$$

(b) From the definition of a linear transformation in the ring $\mathbb{P}$, we have:

$$\mathscr{A}(-\alpha_1) = \mathscr{A}(-1 \cdot \alpha_1) = (-1) \cdot \mathscr{A}(\alpha_1) = -\mathscr{A}(\alpha_1). \tag{30}$$

(c) From the definition of a linear transformation in the ring $\mathbb{P}$, we have:

$$\begin{aligned} \mathscr{A}(\eta) &= \mathscr{A}(\sum_{i=1}^{n} \beta_i \alpha_i) \\ &= \mathscr{A}(\sum_{i=1}^{n} c_i \alpha_i h^+ + \sum_{i=1}^{n} d_i \alpha_i h^-) \\ &= \sum_{i=1}^{n} c_i h^+ \mathscr{A}(\alpha_i) + \sum_{i=1}^{n} d_i h^- \mathscr{A}(\alpha_i) \\ &= \sum_{i=1}^{n} (c_i h^+ + d_i h^-) \mathscr{A}(\alpha_i). \\ &= \sum_{i=1}^{n} \beta_i \mathscr{A}(\alpha_i). \end{aligned} \tag{31}$$

3.2.3 Proof of the Existence Theorem for Linear Transformations in the Ring $\mathbb{P}$

Theory 3.1.2 Let V and V' be vector spaces in the ring $\mathbb{P}$ with V being finite-dimensional. Choose a basis $\alpha_1, \alpha_2, \cdots\cdots, \alpha_n$ for V, and select n vectors $\gamma_1, \gamma_2, \cdots\cdots, \gamma_n$ in V' (which may be identical).

Define:

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$$\mathscr{A}: V \to V', \tag{32}$$

$$\alpha = \sum_{i=1}^{n} \beta_i \alpha_i \mapsto \sum_{i=1}^{n} \beta_i \gamma_i, \beta_i = c_i h^+ + d_i h^- \in \mathbb{P}, (i = 1, 2, 3, \cdots, n). \tag{33}$$

Then $\mathscr{A}$ is a linear mapping from V to V', satisfying furthermore, the linear mapping from V to V' that satisfies is unique.

Proof:

Since the expression of α' as a linear combination of the basis vectors $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is unique, $\mathscr{A}$ is a mapping from V to V'.From equation (40), it immediately follows that $\mathscr{A}(\alpha_i) = \gamma_i$,where $\gamma_i \in V'$.

First, prove that the mapping $\mathscr{A}$ preserves addition:

Let α' and α'' be two arbitrary vectors in the linear space V .Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a basis of V over the ring $\mathbb{P}$ .Then, there exist $\beta_i' = c_i' h^+ + d_i' h^- \in \mathbb{P}$ and $\beta_i'' = c_i'' h^+ + d_i'' h^- \in \mathbb{P}$ such that $\alpha' = \sum_{i=1}^{n} \beta_i' \alpha_i$ and $\alpha'' = \sum_{i=1}^{n} \beta_i'' \alpha_i$, $(i = 1, 2, 3, \cdots, n)$.

Then we have:

$$\begin{aligned}
\mathscr{A}(\alpha' + \alpha'') &= \mathscr{A}(\sum_{i=1}^{n} \beta_i' \alpha_i + \sum_{i=1}^{n} \beta_i'' \alpha_i) \\
&= \mathscr{A}(\sum_{i=1}^{n} (\beta_i' + \beta_i'') \alpha_i) \\
&= \mathscr{A}(\sum_{i=1}^{n} (c_i' h^+ + d_i' h^- + c_i'' h^+ + d_i'' h^-) \alpha_i) \\
&= \mathscr{A}(\sum_{i=1}^{n} [(c_i' + c_i'') h^+ + (d_i' + d_i'') h^-] \alpha_i) \\
&= \sum_{i=1}^{n} [(c_i' + c_i'') h^+ + (d_i' + d_i'') h^-] \gamma_i \\
&= \sum_{i=1}^{n} [(c_i' h^+ + d_i' h^-) + (c_i'' h^+ + d_i'' h^-)] \mathscr{A}(\alpha_i) \\
&= \sum_{i=1}^{n} (\beta_i' + \beta_i'') \mathscr{A}(\alpha_i) \\
&= \sum_{i=1}^{n} \beta_i' \mathscr{A}(\alpha_i) + \sum_{i=1}^{n} \beta_i'' \mathscr{A}(\alpha_i) \\
&= \mathscr{A}(\alpha') + \mathscr{A}(\alpha'').
\end{aligned} \tag{34}$$

That is, the mapping $\mathscr{A}$ still preserves addition over the ring $\mathbb{P}$.

Next, we proceed to prove that the mapping $\mathscr{A}$ preserves scalar multiplication.

Let $\beta = ch^+ + dh^-$ be an arbitrary element in the ring $\mathbb{P}$, and let $\alpha = \sum_{i=1}^{n} \beta_i \alpha_i$ be an arbitrary vector in the vector space V . Then, we have:

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$$\begin{aligned}
\mathscr{A}(\beta\alpha) &= \mathscr{A}((ch^{+}+dh^{-})(\sum_{i=1}^{n}\beta_i\alpha_i)) \\
&= \mathscr{A}((ch^{+}+dh^{-})[\sum_{i=1}^{n}(c_ih^{+}+d_ih^{-})\alpha_i]) \\
&= \mathscr{A}((ch^{+}+dh^{-})(\sum_{i=1}^{n}c_ih^{+}\alpha_i+\sum_{i=1}^{n}d_ih^{-}\alpha_i)) \\
&= \mathscr{A}((ch^{+}+dh^{-})(\sum_{i=1}^{n}c_ih^{+}\alpha_i+\sum_{i=1}^{n}d_ih^{-}\alpha_i)) \\
&= \mathscr{A}(\sum_{i=1}^{n}cc_ih^{+}\alpha_i+\sum_{i=1}^{n}dd_ih^{-}\alpha_i)) \\
&= \mathscr{A}(\sum_{i=1}^{n}(cc_ih^{+}+dd_ih^{-})\alpha_i)) \\
&= \mathscr{A}(\sum_{i=1}^{n}\beta\beta_i\alpha_i) \\
&= \sum_{i=1}^{n}\beta\beta_i\gamma_i \\
&= \beta\sum_{i=1}^{n}\beta_i\gamma_i \qquad (35) \\
&= \beta\mathscr{A}(\alpha).
\end{aligned}$$

That is, the mapping $\mathscr{A}$ still preserves scalar multiplication in the ring $\mathbb{P}$.

Thus, it has been proven that the mapping $\mathscr{A}$ is a linear mapping from V to V'.

Finally, since a linear mapping from V to V' is completely determined by its action on a basis of V, the linear mapping from V to V' satisfying $\mathscr{A}(\alpha_i)=\gamma_i$, $(i=1,2,3,\cdots,n)$ is unique.

4. Conclusions

In conclusion, this paper explores vector spaces defined over the commutative ring $\mathbb{P}$, which is generated by two real numbers and contains zero divisors. By systematically constructing vector spaces over $\mathbb{P}$, this study delves deeper into the algebraic framework and provides a comprehensive analysis of linear transformations and mappings defined within this context. The research not only establishes the foundational principles for the existence and behavior of such mappings but also examines their structural properties and interactions, thereby offering a rigorous framework for further theoretical exploration and application in algebraic systems involving $\mathbb{P}$. Utilizing the unique factorization property of zero divisors in $\mathbb{P}$, the paper establishes an existence theorem for linear mappings, significantly advancing the understanding of the interplay between vector spaces and the structural properties of $\mathbb{P}$. The findings not only enhance the theoretical understanding of algebraic structures related to $\mathbb{P}$, but also open new avenues for further investigations into the broader implications of these structures. This work provides a fundamental perspective for future research in abstract algebra and its potential applications.

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