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Some New Constructions of Centralized Coded Caching Scheme

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Abstract: *As an effective technique to reduce network congestion during peak traffic times, coded caching is being widely studied in wireless network. The coded caching scheme was first proposed by Maddah-Ali and Niesen in 2014. In the placement phase of off- peak times, each file is divided into F equal packets,and each user caches some packets of each file elaborately from the server. In the delivery phase of peak times, each user first requires a file from the server. And then according to each user's cache, the server sends a coded signal with size at most R to users so that various user demands are satisfied.In order to study this problem better, the placement delivery array was defined by Yan Q et al. in 2016. And they found that the problem of designing a coded caching scheme can be converted into the problem of constructing an appropriate PDA.In this paper, we will construct three kinds of new PDAs based on t -design of k = t + 1 and the incidence matrix of a resolvable transversal design and the partitioning of the incidence matrix. Based on these PDAs, we present several new constructions of coded caching schemes, which have obvious advantages in packet number. The results of this paper enrich the constructions of coded caching schemes of low subpacketization level.*

Keywords: *Placement delivery array, coded caching, t-design, transversal design*

1. Introduction

Driven by broadband services such as video on demand and TV chase , it is expected for wireless data transmission to increase significantly in the next few years .In daily life , video transmission has become a major driver in wireless data traffic , and the demand for video transmission is also growing dramatically .Especially when the allocated spectrum resources are limited , the current wireless architecture cannot effectively support the increasing amount of transmission , which will certainly lead to congestion during peak times .

As an effective solution to reduce the pressure of data transmission in peak time, caching technology in wireless networks is being widely studied. And it is thought to be a disruptive technology that can overcome the huge growth of wireless transmission over the next 5G of cellular networks[1].

Maddah-Ali and Niesen first proposed a coded caching scheme in literature[2] in 2014. during the non-peak period placement phase: each file is split into F equal packets, each user caches some packets of each file carefully from the server. During the peak delivery phase: each user first requests a file from the server, but according to the cache of each user, the server sends a coded signal of up to R size to the user. So that the needs of different users are met.

To better study the design of coded caching schemes, the concept of Placement Delivery Array (PDA) was first proposed in[3] in 2016 to describe a kind of cache schemes. Essentially, PDA is an array of $F \times K$ composed of special symbols " * " and some nonnegative integers, In this array, The $k(k \in [0, K-1])$ column represents the $j(j \in [0, F-1])$ column represents the first packet of all files, and states that a PDA can describe the coded caching scheme as follows.

During the content placement phase, the symbol " * " in column k at PDA row j represents the j packet where the user *k* stores all files .

During the content delivery phase, packets requested by different users are represented by the same integer in each row and sent by the server at the same time in the XOR operation (XOR).

Therefore, *F*-division caching scheme for any (K, M, N) caching system can be M/N by a (K, F, Z, S)PDA realization. A relationship between the placement delivery array and the

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(K, M, N) cache system is thus established. As a result, the problem of designing a coded caching scheme can be transformed into the problem of constructing a suitable PDA.

Although optimal in rate, the caching scheme in[2] has its limitation in practical implementations: By this caching scheme, each file is divided into $F = \binom{K}{KM/N}$ packets (The number F is also referred to as the file size or subpacketization in some literature.), which grows exponentially with K in[5]. For practical application, it is important to construct coded caching scheme with smaller packet number.

The following is the research results about placement delivery array.

Placement delivery array (PDA) was introduced in[3] and was shown to be an efficient framework for constructing coded caching schemes. In[3], the Ali-Niesen scheme is proved to correspond to $(t+1)-\left(K,\binom{K}{t},\binom{K-1}{t-1},\binom{K}{t+1}\right)$ PDA (recorded as A-N PDA),and the optimality of Ali-Niesen scheme is proved. At the end of the paper, the new construction of two kinds of PDA $M/N=1/q$ and $M/N=1-1/q$ is given.

Theorem $1^{[3]}$. For every $q, m \in N^+, q \geq 2$, there exists an $(m+1)-(q(m+1), q^m, q^{m-1}, q^{m+1}-q^m)$ PDA.

Theorem $2^{[3]}$. For every $q, m \in N^+, q \geq 2$, there exists an $(q-1)(m+1)-(q(m+1), (q-1)q^m, (q-1)^2 q^{m-1}, q^m)$

In 2016, Yan et al in[4] first connected the construction of PDA to the strong edge coloring problem of a bipartite graph first proving that a PDA and a bipartite strong edge color equivalent, but using the known bipartite graph structure to obtain a new PDA of inclusion Theorem 3 and A-NPDA.

In 2017, Cheng et al put forward the concept of optimal PDA in[6|], Based on A-N PDA, Cheng et al give two kinds of recursive construction of PDA, The PDA of a class of $Z=1$ and $Z=F-1$ for arbitrary positive integer K and F, and the PDA. of some $Z=F-2$ and $Z=F-3$ are different from the previous methods, they put forward a new method to describe PDA with the set of three-dimensional vectors in[12]. This method is used to construct two kinds of PDA lower rate Rand division number F.

Coded caching schemes based on resolvable combinatorial designs from certain linear block codes were studied in[7], and coded caching schemes based on projective geometries over finite fields were reported in[8],[9]. Coded caching schemes from some other block designs, including balanced incomplete block designs (BIBDs), t-designs and transversal designs (TDs), are obtained in[10]. Summaries of known centralized coded caching schemes can be found in[5], [8] and[11].

This paper focuses on some new constructions of PDA in coded caching schemes. Three new PDAs are constructed using t - design and resolvable transversal design. Our contributions are as follows:

Construction 1 gives the method of constructing PDA with t - Design of $k=t+1$: if there is a simple t - $(v, t+1, \lambda)$ design, then there hava a $g-\left(b,\binom{v}{t},\binom{v}{t}-t-1, v\right)$ PDA and a $g-\left(\binom{v}{t}, b, b-\lambda, v\right)$ PDA, where $g=\frac{\lambda}{v}\binom{v}{t}, b=\frac{\lambda}{t+1}\binom{v}{t}$.

Constructing 2 gives the process of constructing PDA using the incidence matrix of a resolvable transversal design, And get the results: if there is a $\mathrm{RTD}\,(n,k)$, then there hava a $n-(n^2, nk, nk-k, nk)$ PDA and a $n-(nk, n^2, n^2-n, nk)$PDA,.where $M/N=\frac{n-1}{n}$.

Construction 3 is given by the block of the incidence matrix of a resolvable transversal design: if there is a $\mathrm{RTD}\,(n,k)$ and a $g-(n,n,Z,S)\mathrm{PDA}$, then there hava a $g-(n^2, nk, Zk, nkS)\mathrm{PDA}$, and some corollaries.

2. Related Concepts of PDA

Definition 1(PDA)[3] . For positive integers K, F, Z and S, an F × K array $P = [p_{j,k}], j \in [0, F), k \in [0, K)$, composed of a specifific symbol "∗ " and S nonnegative integers array (PDA) if it satisfifies the following conditions:

C1. The symbol " ∗ " appears Z times in each column;

C2. Each integer occurs at least once in the array;

C3. For any two distinct entries $p_{j_1 k_1}$ and $p_{j_2 k_2}, p_{j_1 k_1} = p_{j_2 k_2} = s$ is an integer only if

a. $j_1 \neq j_2, k_1 \neq k_2$, i.e., they lie in distinct rows and distinct columns; and

b. $p_{j_1 k_2} = p_{j_2 k_1} = *$, i.e., the corresponding 2 × 2 subarray formed by rows j1, j2 and columns k1, k2 must be of the following form

$$\begin{bmatrix} s & * \\ * & s \end{bmatrix} \text{ or } \begin{bmatrix} * & s \\ s & * \end{bmatrix} \tag{1}$$

Based on a (K, F, Z, S) PDA, P = $P = [p_{j,k}]$ with j ∈[0, F) and k ∈[0, K), an F -division caching scheme for a (K, M, N) caching system with M/N = Z/F can be obtained as follows:

1.Placement Phase: All the files are cached in the same manner. Each file Wi is split into F packets, i.e. $W_i = \{W_{i,j} \quad j \in [0, F)\}, \forall i \in [0, N)$, so that user k ∈ K caches packets

$$C_k = \{W_{i,j}: p_{j,k} = * , \forall i \in [0, N)\} \tag{2}$$

By C1, each user stores N ·Z packets. Since each packet has size 1/F, the whole size of cache is N Z ·1/F = M, which satisfies the users' cache constraint.

2.Delivery Phase: Once the server receives the request $\mathrm{d} = (d_0, d_1, \cdots, d_{K-1})$, at the time slot s, 0 ≤ s < S, it sends

$$\bigoplus_{p_{j,k}=s, j\in[0,F), k\in[0,K)} W_{d_k,j} \tag{3}$$

Illustrative examples of PDA and details for constructing caching schemes from PDAs can be found in[3].

Definition 2. An array P is said to be a g-regular (K, F, Z, S) PDA, g - (K, F, Z, S) PDA or g-PDA for short,if it satisfies C1, C3, and C20 : Each integer appears g times in P where g is a constant.

Theorem 1.1[3] : For any given (K, F, Q, S) PDA, $P = [p_{j,k}]_{F\times K}$, there exists an F -division caching scheme for any (K, M, N) caching system with $\frac{M}{N} = \frac{Q}{F}$ and rate $R = \frac{S}{F}$. Precisely, each user is able to decode its requested file correctly for any demand.

Theorem 1.1 establishes a the connection between a (K, F, Z, S) PDA and a (K, M, N) caching schemes of Fdivision. And the design of the cache scheme is transformed into the problem of constructing a suitable PDA.

Theorem 1.2[3] . For a (K, M, N) cache system, where M/N ∈ {0, 1/K, 2/K, · · ·, 1}, let t = KM/N, then the Ali Niessen scheme corresponds to a $(t+1) - (K, F, Z, S)$ PDA, where $F = \binom{K}{t}, Z = \binom{K-1}{t-1}) \; S = \binom{K}{t+1}$.

Theorem 1.3[6] . For every positive integer F > 10 and 5 - F, there exists an optimal (F, F, F − 3, 6) PDA.

Theorem 1.4[6] . For every positive integer F ≥ 2 and 3 | F, there exists an optimal (F, F, F − 2, 3)PDA.

Theorem 1.5[6] . For every positive integer F ≥ 2 and 3 - F, there exists an optimal (F, F, F − 2, 4) PDA.

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Theorem 1.6[6] . For every positive integer K, F, Z and S, where Z < F, let P is a (K, F, Z, S)PDA, there exists a (K, S, S − (F − Z), F)PDA.

3. The Construction of PDA based on t-Design

Definition 3 (t -design)[13] : Let v, k, λ and t be positive integers such that v > k ≥ t. A t (v, k, λ) -design is a design (X, B) satisfying the following conditions:

- (i) |X| = v;
- (ii) Each block is a k-subset of X;
- (iii) Every t -subset of X is contained in exactly λ blocks.

Notation 1. If the t- design does not contain repeated blocks, then we call it a simple t- design. Without special instructions, the designs mentioned in this chapter are simple t- designs.

Notation 2. 2 − (v, k, λ) also known as balanced incom□plete block designs,which we abbreviate to (v, k, λ)-BIBD.A (v, 3, λ)-BIBD is called a Steiner triple system of order v, or STS(v),a 3 − (v, 4, 1)-design is known as a Steiner quadruple system of order v and is denoted SQS(v).

Theorem 3.1[13] . A t − (v, k, λ) design has five parameters v, b, r, k and λ, where v is called the order of the design, and r is called repeated number, λ is called t- balanced number, k is called block capacity, and b is called block number, and the parameters satisfy the following relation:

$$b\binom{v}{t}=\lambda\binom{v-1}{t-1},r\binom{k-1}{t-1}=\lambda\binom{v-1}{t-1} \tag{4}$$

Theorem 3.2[14] .There exists an SQS(v) ,iff n ≥ 3 and v ≡ 1 or 3(mod6).

Theorem 3.3[14] . For all integers n ≥ 2,there exists SQS(2n).

Theorem 3.4[15] . Let v ≥ t + 1 and $v\equiv(\mathrm{mod}((t+1)!)^{2t+1})$ then there exists a simple $t-(v,t+1,((t+1)!)^{2t+1}$-design.

Next we construct a kind of g-PDA with t-design of k = t + 1.

Construction 1. Let D = (X, B) is a simple t−(v, t+1, λ) -design, where X = {1, 2, · · ·, v}, B = {Bj | j ∈[1, b]}, and $b=\frac{\lambda}{t+1}\binom{v}{t}$ Let T denote a t-subsets of X, let T denote the set of all a t-subsets of X, that is

$$\mathcal{T}=\left\{T_i|i\in\left[1,\binom{v}{t}\right]\right\} \tag{5}$$

Construct an array $D=[d_{ij}],i\in\left[1,\binom{v}{t}\right)\Big],j\in[1,b]$ we can get a $\binom{v}{t}\times b$ array. Where

$$d_{ij}=\begin{cases}*, & \text{if } T_i\not\subseteq B_j\\ B_j-T_i, & \text{if } T_i\subseteq B_j\end{cases} \tag{6}$$

for every x ∈ X,

$$p_{ij}=\begin{cases}*, & \text{if } d_{ij}=*\\ x-1, & \text{if } d_{ij}=\{x\}\end{cases} \tag{7}$$

then array $P=[p_{ij}],i\in\left[1,\binom{v}{t}\right],j\in[1,b]$ is a $g-\left(b,\binom{v}{t},\binom{v}{t}-t-1,v\right)$ PDA, where $g=\frac{(t+1)b}{v}$

Theorem 3.5. If there exists a simple t−(v, t+1, λ) design, then there exists a $g-\left(b,\binom{v}{t},\binom{v}{t}-t-1,v\right.$) PDA, where $g=\frac{\lambda}{v}\binom{v}{t},b=\frac{\lambda}{t+1}\binom{v}{t}$. rate $R=\frac{v}{\binom{v}{t}}$.

Proof: Clearly , P =[pij] is a $\binom{v}{t}\times b$ array, so $K=b=\frac{\lambda}{t+1}\binom{v}{t},F=\binom{v}{t}$ By the construction 1, we have $S=|X|=v$ We can prove tha P =[pij] satisfy definition of g -PDA.

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We first prove that P satisfies C2 as following, that is the symbol " ∗ " appears exactly Z times in every column.

Choosing any column of array P, let's it mark j-th column of block Bj .When $T_i \not\subseteq B_j, p_{ij} = *$. Because |Bj | = t + 1 and Bj contains $\binom{t+1}{t}$ t t-subsets, so $\binom{v}{t} - \binom{t+1}{t}$ t-subsets are not included in Bj , that is " ∗ " appears $\binom{v}{t} - \binom{t+1}{t}$ times in this column . So array P satisfies C1 and

$$Z = \binom{v}{t} - \binom{t+1}{t} = \binom{v}{t} - t - 1 \tag{8}$$

Second, we will prove that P satisfies C20 , that is, every integer appears g times in the array.

For every integer s ∈[0, v − 1], let's s = s0 − 1, then s0 ∈[1, v], this is s0 ∈ X. By the definition of t - design, each point appears in r blocks, So s appears r times in the array, P satisfies C20 and

$$g = r = \frac{\lambda}{v}\binom{v}{t} \tag{9}$$

We will prove P satisfies C3, for any two different pi1,j1 and pi2,j2 , where pi1,j1 = pi2,j2 = s is an integer, All of them i1 ≠ i2, j1 ≠ j2 and pi1,j2 = pi2,j1 = ∗ .

First we prove the array P satisfies C3-a.

Because pi1,j1 and pi2,j2 are different, then they may be in the same row , or in a different row , or different columns in different rows.

Suppose pi1,j1 and pi2,j2 are in the same row, that is i1 ≠ i2 and j1 = j2 = j, pi1,j = pi2,j = s. Since i1 ≠ i2, then we have Ti1 ≠ Ti2 . Due to pi1,j = pi2,j = s, then Ti1 ∪ {s+ 1} = Ti2 ∪ {s+ 1} = Bj . Since |Ti1 | = |Ti2 | = t, |Bj | = t+ 1, then we have Ti1 = Ti2 and Ti1 = Ti2 ,which is a contradiction. So they cannot be in the same column in different rows.

If pi1,j1 and pi2,j2 are in the same row of different columns, that is i1 = i2 = i and j1 ≠ j2, that is pi,j1 = pi,j2 = s. Due to j1 ≠ j2, there is Bj1 ≠ Bj2 .Since pi,j1 = pi,j2 = s, then Ti ∪ {s + 1} = Bj1 = Bj2 . Since In construction 1, t-design has no repeat block, so Bj1 = Bj2 and Bj1 ≠ Bj2 ,which is a contradiction. So they can't be in the same row of different columns.In conclusion, if pi1,j1 = pi2,j2 = s is an integer , they may only be in different columns in rows, that is i1 ≠ i2, j1 ≠ j2. So P satisfies C3-a.

We prove the array P satisfies C3-b.

pi1,j1 = pi2,j2 = s are known to be integers, and i1 ≠ i2, j1 ≠ j2. From construction 1 s+1 /∈ Ti1 , s+1 /∈ Ti2 . Since Ti1 ≠ Ti2 , and |Ti1 | = |Ti2 | = t, then there exists a ∈ Ti2 and a /∈ Ti1 .Since Ti1 ∪ {s + 1} = Bj1 , then a /∈ Bj1 . So Ti2Bj1 , pi2,j1 = ∗ . Similarly, pi1,j2 = ∗ . so P satisfies C3-b.In conclusion, $P = [p_{ij}]$ is a $g - \left(b, \binom{v}{t}, \binom{v}{t} - t - 1, v\right)$PDA,where $g = \frac{\lambda}{v}\binom{v}{t}, b = \frac{\lambda}{t+1}\binom{v}{t}$.

The definition of t- design shows that for every Ti ∈ T appears exactly in λ blocks, that is ,"∗ " in the array P appears b − λ times in each row, so the transpose of the array is a g-PDA, and there is the following corollaries.

Corollary 3.6. If there exists a simple t−(v, t+1, λ) design, then there exists a $g - \left(\binom{v}{t}, b, b - \lambda, v\right)$ PDA, where $g = \frac{\lambda}{v}\binom{v}{t}, b = \frac{\lambda}{t+1}\binom{v}{t}$, rate $R = \frac{v}{b}$.

Example 3.1. Let (X, B) is a 2- (5, 3, 3)-design.

$X = \{1,2,3,4,5\}$
$\mathcal{B} = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},$
$\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\},$
$\mathcal{T} = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},$
$\{3,5\},\{4,5\}\}$

A 6 - $(10,10,7,5)$ PDA can be obtained from construction 1

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$$P=\begin{bmatrix} 2 & 3 & 4 & * & * & * & * & * & * & * \\ 1 & * & * & 3 & 4 & * & * & * & * & * \\ * & 1 & * & 2 & * & 4 & * & * & * & * \\ * & * & 1 & * & 2 & 3 & * & * & * & * \\ 0 & * & * & * & * & * & 3 & 4 & * & * \\ * & 0 & * & * & * & * & 2 & * & 4 & * \\ * & * & 0 & * & * & * & * & 2 & 3 & * \\ * & * & * & 0 & * & * & 1 & * & * & 4 \\ * & * & * & * & 0 & * & * & 1 & * & 3 \\ * & * & * & * & * & 0 & * & * & 1 & 2 \end{bmatrix}$$

Table 1 shows that the PDA obtained by construction 1 is F (division number) effectively reduced compared with the Ali-Niesen cache scheme under the same cache system.

Table 1: A Construction of PDA example 3.1 AND A − NPDA.

$\left(K, \frac{M}{N}\right)$	parameter	Ali-Niesen	example 3.1
$\left(10, \frac{7}{10}\right)$	R	0.375	0.5
	F	120	10

For the existence of t − (v, t + 1, λ)-designs, we give some simple t- design of k = t + 1. For example, when t = 2, λ = 1, we can get from the theorem of 3.2, when v ≥ 3 and v ≡ 1or 3(mod6), there exists STS(v). Therefore, we have the following Corollary.

Corollary 3.7. When v ≥ 3 and v ≡ 1 or 3(mod6) ,there exists $\frac{v-1}{2}-\left(\frac{v(v-1)}{6},\frac{v(v-1)}{2},\frac{v(v-1)-6}{2},v\right)$ PDA, where rate is $R=\frac{2}{v-1}$.

Similarly, when t = 3, λ = 1 , from Theorem 3.3, thereexists SQS(2n). Thus , there is the following Corollary.

Corollary 3.8. When n ≥ 2 , there exists $\frac{(2^n-1)(2^{n-1}-1)}{3}$ $\left(\frac{2^{n-2}(2^n-1)(2^{n-1}-1)}{3},\frac{2^n(2^n-1)(2^{n-1}-1)}{3},\frac{2^n(2^n-1)(2^{n-1}-1)}{3}-4,2^n\right)$ PDA, where rate For the existence of $t-(v,t+1,\lambda)$ -designs, we give some simple t - design of $k=t+1$. For example, when $t=2,\lambda=$1, we can get from the theorem 3.2, when $v\geq 3$ and $v\equiv 1$ or $3(\mathrm{mod}6)$, there exists STS (v). Therefore, we have the following Corollary.

Corollary 3.7. When $v\geq 3$ and $v\equiv 1$ or $3(\mathrm{mod}6)$,there exists $\frac{v-1}{2}-\left(\frac{v(v-1)}{6},\frac{v(v-1)}{2},\frac{v(v-1)-6}{2},v\right)$ PDA, where rate is $R=\frac{2}{v-1}$.

Similarly, when $t=3,\lambda=1$, from Theorem 3.3, there exists SQS (2^n). Thus, there is the following corollary.

Corollary3.8. When $n\geq 2$, there exists $\frac{(2^n-1)(2^{n-1}-1)}{3}$ $\left(\frac{2^{n-2}(2^n-1)(2^{n-1}-1)}{3},\frac{2^n(2^n-1)(2^{n-1}-1)}{3},\frac{2^n(2^n-1)(2^{n-1}-1)}{3}-4,2^n\right)$PDA, where rate $R=\frac{3}{(2^n-1)(2^{n-1}-1)}$.

Example 3.2.Let (X, B) is a STS(7),

$X=\{1,2,3,4,5,6,7\}$

B = {{1, 2, 3}, {1, 4, 5}, {1, 6, 7}, {2, 4, 7}, {2, 5, 6}, {3, 4, 6}, {3, 5, 7}}.A 3-(7,21,18,7) PDA can be obtained from construction 1.

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$$D = \left[\begin{array}{c|ccccccc}
 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \\ \hline
T_1 & 3 & * & * & * & * & * & * \\
T_2 & 2 & * & * & * & * & * & * \\
T_3 & * & 5 & * & * & * & * & * \\
T_4 & * & 4 & * & * & * & * & * \\
T_5 & * & * & 7 & * & * & * & * \\
T_6 & * & * & 6 & * & * & * & * \\
T_7 & 1 & * & * & * & * & * & * \\
T_8 & * & * & * & 7 & * & * & * \\
T_9 & * & * & * & * & 6 & * & * \\
T_{10} & * & * & * & * & 5 & * & * \\
T_{11} & * & * & * & 4 & * & * & * \\
T_{12} & * & * & * & * & * & 6 & * \\
T_{13} & * & * & * & * & * & * & 7 \\
T_{14} & * & * & * & * & * & 4 & * \\
T_{15} & * & * & * & * & * & * & 5 \\
T_{16} & * & 1 & * & * & * & * & * \\
T_{17} & * & * & * & * & * & 3 & * \\
T_{18} & * & * & * & 2 & * & * & * \\
T_{19} & * & * & * & * & 2 & * & * \\
T_{20} & * & * & * & * & * & * & 3 \\
T_{21} & * & * & 1 & * & * & * & *
\end{array}\right]$$

A 3-(21,7,6,7)PDA can be obtained from corollary 3.6. As shown in Table 3.2, the corresponding cache scheme of the PDA is F effectively reduced compared with the Ali-Niesen cache scheme under the same cache system. We can get λ very large the t- design of k = t + 1 from theorem 3.4.Thus, there is the following corollary.

Corollary 3.9. When v ≥ t + 1 and $v \equiv t(\mathrm{mod}((t+1)!\,)^{2t+1})$, then there exists a $g-\left(b,\binom{v}{t},\binom{v}{t}-t-1,v\right)$ PDA, where $g=\frac{\lambda}{v}\binom{v}{t}, b=\frac{\lambda}{t+1}\binom{v}{t}, \lambda=((t+1)!\,)^{2t+1}$, where rate is $R=\frac{v}{\binom{v}{t}}$

4. The Constructions of PDA Based on Transversal Design

Def inition 4[13] . Let k ≥ 2 and n ≥ 1. A transversal design TD(k, n) is a triple (X, G, B) such that the following properties are satisfied:

(1) X is a set of kn elements called points,

(2) G is a partition of X into k subsets of size n called groups,

(3) B is a set of k -subsets of X called blocks,

(4) any group and any block contain exactly one common point, and

(5) every pair of points from distinct groups is contained in exactly one block.

Notation 3. That the “groups” in a transversal design are just subsets of points; they are not algebraic groups. Also, a TD (2, n) exists trivially for all integers

If B can be divided into n parallel classes πα(α ∈ I), such that for B, B0belongs to the same parallel class Πα, there are B ∩B0 = ∅ and ∪Bi∈Πα Bi = X, then triple (X, }, B)is called resolvable transversal design RTD(n, k).

The following gives the incidence matrix of RTD (n, k) , M is a 0-1 matrix of v ×b, which can be obtained from the condition (4),(5).

There are n2 blocks, where v = |x| = nk, B = |B| = n2,

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$$M = (m_{ij})_{1 \le i \le v, 1 \le j \le b'} \tag{10}$$

where point p1, p2, ···, pvtag line, block B1, B2, ···, Bbtag column, and

$$m_{ij} = \begin{cases} 1, & p_i \in B_j \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

Construction 2, there are k groups in RTD(n, k) denoted as G = {G0, G1, · · · , Gk−1}, and B has n parallel classes, de-noted as P0,P1, · · · ,Pn−1. Now we use the incidence matrix M of RTD(n, k) to construct a new PDA, and let the array be A =[aij]v×b , where 1 ≤ i ≤ v = nk, 1 ≤ j ≤ b = n2.

$$a_{ij} = \begin{cases} (x, y), \text{ if } m_{ij} = 1, i \in G_x, B_j \in \mathcal{P}_y \\ \quad *, \text{ otherwise } (\text{ that is } m_{ij} = 0) \end{cases} \tag{12}$$

Where 0 ≤ x ≤ k − 1, 0 ≤ y ≤ n − 1.Let s = nx + y, then S = {0, 1, · · · , nk − 1},and

$$p_{ij} = \begin{cases} s = nx + y, & \text{if } a_{ij} = (x, y) \\ \quad *, & \text{otherwise} \end{cases} \tag{13}$$

then array P =[pij] is a $n-(n^2, nk, nk-k, nk)$ PDA.

Theorem 4.0: If there is a RTD(n, k), then there exists $n-\ (n^2, nk, nk-k, nk)$ PDA and $n-(nk, n^2, n^2-n, nk)$ PDA , their coded caching scheme have $M/N = \frac{n-1}{n}$ and the rates are 1 and $\frac{k}{n}$.

Proof: From construction 2, we can get that P is a nk×n2 array, then K = n2 , F = nk. We prove that array P satisfies the definition of g -PDA .

First, we prove that the array P satisfies C1.

By the definition of transversal design, every block contains k points, then 1 appears k times in each column of the incidence matrix M , 0 appears nk − k times. From construction 2, if mij = 0, then pij = ∗ . Thus " ∗ " appears nk − k times in each column in the array P.

Second, we prove that P satisfies C2'.

For every s ∈[0, nk − 1] . By the construction 2, we have s = nx + y, that is mij = 1, and i ∈ Gx, Bj ∈ Py. Group Gx has n points, Py has n blocks, and every group and block contains a common point. So there are n (x, y) in the A . Since s = nx + y, then s appears n times in the array P.

Third, we prove that P satisfies C3.

For any pi1,j1 = pi2,j2 = s in array P , from construction 2, ai1,j1 = ai2,j2 = (x, y) in A, that is i1, i2 ∈ Gx, blocks Bj1 , Bj2 ∈ Py. Since pi1,j1 and pi2,j2 is different, and every group and block contains exactly one common point,if i1 = i2, then Bj1 = Bj2 , that is j1 = j2. Similarly, if j1 = j2, then i1 = i2. Since pi1,j1 and pi2,j2 is different, so,i1 ≠ i2 and j1 ≠ j2.

Due to every group and block contains exactly one common point, so group Gx and block Bj1 only one common point i1. Since i2 ∈ Gx, so i2 /∈ Bj1 . So there's mi2,j1 = 0 in the correlation matrix M , there is pi2,j1 = ∗ in P. Similarly, pi1,j2 = ∗ .

In conclusion, P is a $n-(n^2, nk, nk-k, nk)$ PDA, and $R = \frac{S}{F} = \frac{k}{n}$.

If let transpose array of P be P^T , we can prove P^T is a $n \quad (nk, n^2, n^2-n, nk)$ PDA.

Construction 3. Let P is a g − (n, n, Z, S)PDA, and we stipulate that the P + S denotes that every item in the array P adds an integer S, that is pi,j + S, and ∗ + S = ∗ . If there exists a RTD(n, k), then there is a P' as shown below, where

$$P' = \begin{pmatrix} P & P+S & \cdots & P+(n-1)S \\ P+nS & P+(n+1)S & \cdots & P+(2n-1)S \\ \vdots & \vdots & & \vdots \\ P+(k-1)nS & P+[(k-1)n+1]S & \cdots & P+(kn-1)S \end{pmatrix}$$

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Theorem 4.1. If there is a RTD(n, k) and a g−(n, n, Z, S) PDA, then there exists $g-(n^2, nk, Zk, nkS)$ PDA

First of all, we prove that the array P is a n×n array. Obviously, P' is a nk ×n2 array.Then we prove P' satisfies the definition of PDA.

First, we prove P' satisfies C1.

Because P is a g − (n, n, Z, S) PDA, then " ∗ " appears Z times for each column of the array P, due to the block array p' can be divided into k row and n column, so " ∗ " appears appears Zk times in each column of array p prime.

Second, we prove P' satisfies C2'.

Because P is a g − (n, n, Z, S)PDA, for every s ∈[0, S − 1], s appeared g times in the P. Similarly, for every s ∈[S, 2S − 1], s appeared g times in P +S . And so on,, for every s ∈[0, nkS −1], s appears g times in P'.

Third, we prove P' satisfies C3.

If s ∈[0, S − 1], then the integer s only appears in p. It is known that P is a g − (n, n, Z, S)PDA, so for any two different items pi1,j1 = pi2,j2 = s, there must be i1 ≠ i2, j1 ≠ j2 and pi2,j1 = pi1,j2 = ∗ . Similarly, if s' ∈[S, 2S − 1], then the integer s' only appears in P +S . Since P is a g−(n, n, Z, S)PDA, so for any two different items pi1,j1 = pi2,j2 = s', there is still i1 ≠ i2, j1 ≠ j2 and pi2,j1 = pi1,j2 = ∗ . And so on, for any two different items pi1,j1 = pi2,j2 = s', where s ∈[0, nkS − 1], we have i1 ≠ i2, j1 ≠ j2 and pi2,j1 = pi1,j2 = ∗ .

In conclusion, P'is a $g-(n^2, nk, Zk, nkS)$ PDA.

It is known that there exists $n-\left(n, n, 1, \frac{n(n-1)}{2}\right)$ PDA. The following corollary can be obtained from theorem 4.1.

Corollary 4.2. If there is a RTD(n, k), then there exists 2 $-\left(n^2, nk, k, \frac{n^2k(n-1)}{2}\right)$ PDA and 2-$\left(nk, n^2, n,, \frac{n^2k(n-1)}{2}\right)$ PDA ,where the transmission rates are $\frac{n(n-1)}{2}$ and $\frac{k(n-1)}{2}$ From theorem 1.6 and theorem 4.1 the following corollary can be obtained.

Corollary 4.3. If there is a g−(n, n, Z, S)PDA and a RT D(n, k), then there exists a $gS-(n^2, nkS, nk(S-1)+Zk, nk)$ PDA.From theorem 1.3 and theorem 4.1 the following corollary can be obtained.

Corollary 4.4. For every positive integer n, k, and n >10, 5 - n, if there exists a RTD(n, k), then there is a $(n^2, nk, (n-3)k, 6nk)$ PDA.The following corollary can be obtained from theorem 1.3 and corollary 4.3.

Corollary 4.5. For every positive integer n, k, and n >10, 5 - n, if there is a RTD(n, k), then there exists a $(n^2, 6nk, 6nk+3k, nk)$ PDA. From theorem 4.1and theorem 1.4 the following inference can be obtained.

Corollary 4.6. For every positive integer n, k, and n > 2, 3 | n, if there is a RTD(n, k), then there exists a $(n^2, nk, (n-2)k, 3nk)$ PDA.From theorem 4.1and theorem 1.5 the following inference can be obtained.

Corollary 4.7. For every positive integer n, k, n > 2, 3 - n, if there is a RTD(n, k), then there exists a $(n^2, nk, (n-2)k, 4nk$) PDA.

Finally, we give an example of RTD(5,2).

Example 4.1. The incidence matrix of RTD(5,2) is given asfollows:

$$M=\begin{bmatrix}
1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1
\end{bmatrix}$$

From construction 2, we can get array P1 is 5−(25, 10, 8, 10)PDA, where

$$P_1=\begin{bmatrix}
0&*&*&*&*&1&*&*&*&*&2&*&*&*&*&3&*&*&*&*&4&*&*&*&*\\
&0&&*&*&*&1&*&*&*&*&2&*&*&*&*&3&*&*&*&*&4&*&*&*\\
&&0&*&*&*&*&1&*&*&*&*&2&*&*&*&*&3&*&*&*&*&4&*&*\\
&&*&0&*&*&*&*&1&*&*&*&*&2&*&*&*&*&3&*&*&*&*&4&*\\
&&*&*&0&*&*&*&*&1&*&*&*&*&2&*&*&*&*&3&*&*&*&*&4\\
5&*&*&*&*&6&*&*&*&*&7&*&*&*&*&8&*&*&*&*&9&*&*&*&*\\
&5&&*&*&*&6&*&*&*&*&7&*&*&*&*&8&*&*&*&*&9&*&*&*\\
&&5&*&*&*&*&6&*&*&*&*&7&*&*&*&*&8&*&*&*&*&9&*&*\\
&&*&5&*&*&*&*&6&*&*&*&*&7&*&*&*&*&8&*&*&*&*&9&*\\
&&*&*&5&*&*&*&*&6&*&*&*&*&7&*&*&*&*&8&*&*&*&*&9
\end{bmatrix}$$

From Corollary 4.2, we can get array P2 is 2 − (25, 10, 2, 100)PDA.

$$P_2=\begin{bmatrix}
&0&1&2&3&&10&11&12&13&*&20&21&22&23&*&30&31&32&33&*&40&41&42&43\\
0&*&4&5&6&10&*&14&15&16&20&*&24&25&26&30&*&34&35&36&40&*&44&45&46\\
1&4&*&7&8&11&14&*&17&18&21&24&*&27&28&31&34&*&37&38&41&44&*&47&48\\
2&5&7&*&9&12&15&17&*&19&22&25&27&*&29&32&35&37&*&39&42&45&47&*&49\\
3&6&8&9&*&13&16&18&19&*&23&26&28&29&*&33&36&38&39&*&43&46&48&49&*\\
&50&51&52&53&&60&61&62&63&*&70&71&72&73&*&80&81&82&83&*&90&91&92&93\\
50&*&54&55&56&60&*&64&65&66&70&*&74&75&76&80&*&84&85&86&90&*&94&95&96\\
51&54&*&57&58&61&64&*&67&68&71&74&*&77&78&81&84&*&87&88&91&94&*&97&98\\
52&55&57&*&59&62&65&67&*&69&72&75&77&*&79&82&85&87&*&89&92&95&97&*&99\\
53&56&58&59&*&63&66&68&69&*&73&76&78&79&*&83&86&88&89&*&93&96&98&99&*
\end{bmatrix}$$

5. Conclusions

We propose a new perspective of constructing a new cache coding scheme based on PDA. Several new types of PDA structures and corresponding encoding and caching schemes are constructed successfully.The main tools used include t-design and cross section design. Using these tools, we get three kinds of constructions and generalize some conclusions. The results of this paper enrich the construction of the centralized cache coding scheme, and have certain application value.

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