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Application of Conditional Extreme Values in Higher Perspective—Take "Conic Curve" Problems as Examples

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Abstract: *With the reform of secondary school mathematics, teaching gradually emphasizes tracing the nature of the problem. Conditional extreme value is one of the knowledge points that bridges higher mathematics and elementary mathematics. This paper adheres to the idea of the high point of view, connects the conditional extreme value in higher mathematics with the most value problem taught in secondary school, and uses Lagrange number multiplication to solve the distance, area extreme value problem. Conic curve as an example, the high point of view of the background of the conditional extreme value solution and high school commonly used to solve the inequality problem ideas. By solving the partial derivatives listed in the system of equations to determine the coordinates of the point, the practice is simple, novel ideas, easy to operate. It is conducive to students' expansion of ideas and enhancement of disciplinary literacy.*

Keywords: *high point of view; conditional extremes; conic curves*

1. Introduction

Higher viewpoints are the use of higher mathematical knowledge methods ideas, etc, to analyze and study elementary mathematical problems[1]. With the introduction of this concept, some teachers will introduce higher mathematics into the classroom during the teaching process. This behavior not only brings new challenges to secondary school mathematics teaching but also brings new ideas when thinking about problem-solving strategies. Functional thinking, calculus theory, and vectors in higher mathematics are very much related to secondary school mathematics. The most valuable problems in secondary school mathematics are the most common ones, involving equations and inequalities, series, analytic geometry, and so on, and students need not only a solid foundation but also flexible ideas and qualified arithmetic ability when solving this part of the problem. Conditional extremes in higher mathematics provide new ideas and methods for such problems. In this paper, we will show that the use of conditional extremes can be used not only to solve problems of inequalities but also for some simple topics of analytic geometry in the category of finding extremes.

2. Fundamentals of conditional extremes

Extreme value problems with conditions attached are called conditional extreme value problems (extreme value problems without constraints are also known as unconditional extreme value problems).The general form of the conditional extreme value problem is to find the extreme value of the objective function $y = f(x_1, x_2, \cdots, x_n)$ under the constraints of the condition set $\varphi_k(x_1, x_2, \cdots, x_n) = 0, k = 1,2,\cdots, m(m > n)$ [2].The method used to solve this type of problem is Lagrange number multiplication. The following is an explanation of a simple binary function as an example.To find the extreme value of the binary function $f(x, y)$ under the constraint $\varphi(x, y) = 0$, the Lagrange number multiplication method is applied and the basic steps are:

① as a Lagrangian function

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$$

ISSN 2616-5805 Vol. 5, Issue 3: 8-15, DOI: 10.25236/AJMS.2024.050302

② List the system of equations

$$\begin{cases} L_x = f_x(x,y) + \lambda\varphi_x(x,y) = 0 \\ L_y = f_y(x,y) + \lambda\varphi_y(x,y) = 0 \\ L_\lambda = \varphi(x,y) = 0 \end{cases}$$

③Solving the above system of equations gives the stationing point (x_0, y_0) .

④ Determine whether the stationary point is a conditional extreme value point according to the specific topic, and if it is not a practical application of the problem, determine it by second-order differentiation.

3. Application of conditional extremes to inequalities

Conditional extreme value is the knowledge in advanced mathematics, mainly used to solve the problem of solving the extreme value of a function under certain constraints, which is ultimately to solve the problem of inequality. There are specific requirements for learning inequalities in high school. The General High School Mathematics Curriculum Standards (2017 edition revised in 2022) require students to master the basic inequalities $2\sqrt{ab} \le a+b(a,b \ge 0)$. With specific examples, they can use basic inequalities to solve simple maximum or minimum value problems[3] . Simple inequality problems can be solved by their properties, and students need to master the basic knowledge and the corresponding problem-solving skills, including some common deflation techniques, while the binary function under constraints to find the most value is a high-frequency test in the high school paper. Common methods for solving binary quadratic inequalities include mean value inequality and elimination, and special types can also be considered by combining numbers and shapes[4]. And the conditional extreme value provides a new idea for solving this kind of problem. Inequality problems in secondary school are all about finding a geometrically meaningful inequality under certain constraints, which is essentially converting the inequality to find the most value problem into solving the most value problem of a binary function or even a multivariate function. The use of Lagrange number multiplication to construct functions generalizes this type of problem into a mathematical model: if the real number $a,b,c,...$ meets $f(a,b,c,...)=0$, the function $g(a,b,c,...)$ of the most value problem[5]. This approach enriches students' thinking, develops their analytical and problem-solving abilities, and examines their core mathematical literacy such as logical reasoning, mathematical operations, and intuitive imagination, which is conducive to the improvement of core mathematical literacy. The class of conditional extremes has a wide range of applications at the high school level and is covered in important knowledge modules such as inequalities, series, functions, and analytic geometry. Conditional extremes are more studied in inequality questions at this stage, and the following article will share this knowledge based on this knowledge in conic curves involving two types of maximum value problems.

4. Applications of conditional extremes to plane analytic geometry

Ellipses, hyperbola, and parabola, this part of the content is the first more difficult plate that students come into contact with when they enter the elective class. Middle school students study primary function, quadratic function, and inverse example function and know that they represent straight lines, parabolas, and hyperbolas respectively. High school students study the image and properties of elementary functions in the first book of the mandatory and have a vague understanding of the relationship between geometry and algebra. Plane analytic geometry in 2018 after the intensive reform of mathematical content, such problems in the college entrance examination began to return to the direction of exploring the analytic pathway of geometric problems. In this context, algebraic analysis became the focus, seeking quantitative relationships in the topic, graphical relationships, and becoming a problem that teachers and students solve together[6]. To master the analysis of geometric problems, students need to be familiar with the basic connections between graphs and quantities, skillfully use tools such as vectors and coordinate arithmetic, accumulate and study basic problem types and classical models, and, of course, most importantly, strengthen algebraic thinking and return to the textbook itself.

Conic Curve is the specific content of plane analytic geometry in high school curriculum, and it is

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also the first time for students to intuitively link the geometry formed by smooth closed curves with algebraic equations. From the point of view of the topic, the centrifugal rate problem, the distance problem, and the area problem are all frequently examined types of questions. The centrifugal rate problem is a more basic test, the centrifugal rate of the value of the problem can use the basic properties of conic curves, combined with the idea of equations to calculate; and the range of the centrifugal rate of the problem can be analyzed with the help of inequality and function ideas[7].Students have basically established a system of equations in junior high school, and are relatively unfamiliar with the connection between curves and functions and inequalities, and the examination of conic curves in the college entrance exams in recent years also reflects the importance of the idea. From the point of view of function, the conic curve is the real exposure of students to a binary quadratic function. For teachers, students are taught to use the language of algebra to describe the characteristics of graphs, to draw images based on the relationships described in the text of the topic when solving problems, to find the corresponding algebraic relationships to construct equations, to use visual images and algebraic operations to get the results, and ultimately to build a system of geometry and functions. But for students, the transition from equations to functions is undoubtedly a shift from the concrete to the abstract, students need to realize the difference between the two from the ideological, in the understanding of equations and functions before they can continue to extend to inequalities, so in the first book of the necessary to start with equations, functions and inequalities, the contents of these content when the transition from junior high school to high school focuses on the often also difficult. For the conic curve in the emergence of extreme value problems, there is more than one way, in addition to using the conventional method, the same can also be used to answer the method of conditional extreme value, the following will give a few examples.

4.1 Applications in Circles and Straight Lines

High school learning plane analytic geometry is in the elective part, the first study is straight line and circle, which is the most important is the equation of a straight line and the equation of a circle. In the right-angled coordinate system of a straight line and a circle is the easiest to form the intersection and tangent, which is the easiest for students to understand the two positional relationships, tangent case there will be a tangent and tangent point, the intersection of the case is the common point of the coordinates, which naturally leads to the equation of a straight line and the circle for association. The equation of a straight line and the equation of a circle are linked. According to the relationship between the position of the line and the circle, we can get the relationship between the distance from the center of the circle to the line and the size of the radius, for example, in the following example.

Example 1: (2023. National B)The real number x, y is known to satisfy $x^2+y^2-4x-2y-4=0$,

find the maximum value of $x-y$?

Solution 1: From $x^2+y^2-4x-2y-4=0$,we get $(x-2)^2+(y-1)^2=9$,which means the circle with center is $(2,1)$ and radius is 3 . Let $x-y=u$,then $x-y-u=0$,from the question the line has a common point with the circle, then $|1-u|\le 3\sqrt{2}$,solve $1-3\sqrt{2}\le u\le 1+3\sqrt{2}$,so the maximum value of $x-y$ is $1+3\sqrt{2}$.

Ans: The title of the equation through the transformation can be obtained by the center of the circle is $(2,1)$ and radius is 3 . The question asks for the most value of $x-y$ .Students in the solution very easily have no direction, and may wish to set out the unknowns, through the observation can be obtained by the equation of the straight line. "Straight line and circle" is most likely to be associated with the positional relationship, through the calculation of the two positional relationships, constitute the corresponding inequality for the answer.

Solution 2: $(x-2)^2+(y-1)^2=9$, so that , $\begin{cases} x=2+3\cos\theta \\ y=1+3\sin\theta \end{cases}$ $0\le\theta\le 2\pi$, so that

$x-y=1+3\sqrt{2}\cos(\theta+\frac{\pi}{4})$, when $\cos(\theta+\frac{\pi}{4})=1$, the maximum value of $x-y$ is $1+3\sqrt{2}$.

ISSN 2616-5805 Vol. 5, Issue 3: 8-15, DOI: 10.25236/AJMS.2024.050302

Ans: In addition to the perspective from the right angle coordinates, the permutation method is also commonly used in mathematical problem-solving ideas, commonly used in the conic curve in a class of permutations is a trigonometric permutation, the use of polar coordinates expressed in the title of the $x-y$, and then through the must learn the trigonometric functions in the auxiliary angle formula can be the same to get the maximum value. This method for senior students need to do is to build their knowledge system framework, and summarize a variety of solutions to a problem. The new textbook on parametric equations and polar coordinates for deletion, as a teacher in the daily teaching should focus on expanding students' ideas, and cultivate students' habit of solving multiple problems, through the accumulation of time to achieve the purpose of a clever solution to a problem.

The above two ideas are common in high school problem solving, the use of the meaning of the equation itself, combined with the positional relationship between the graphs, listed the corresponding inequality for the answer. This way is more embodied in the combination of mathematical and morphological ideas, to develop the students of geometry and algebra between the logical reasoning ability. The use of letters to represent the most value of the problem is essentially the construction of inequalities, listed inequalities should start to solve the answer. For the most value problem, the following is the conditional extreme value of the solution.

Solution 3: Let $f(x,y)=x-y$, $\varphi(x,y)=x^2+y^2-4x-2y-4$ The problem is transformed into a problem of finding the maximum value of the function under the condition $\varphi(x,y)$ Find the maximum value of the function $f(x,y)=x-y$ under the Construct the Lagrangian:

$$L(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y),$$

solve systems of equations

$$\begin{cases} L_x = 1+\lambda(2x-4)=0 \\ L_y = -1+\lambda(2y-2)=0 \\ L_\lambda = x^2+y^2-4x-2y-4=0 \end{cases},$$

catch (a disease)

$$\begin{cases} x_1 = \dfrac{3\sqrt{2}}{2}+2 \\ x_2 = 2-\dfrac{3\sqrt{2}}{2} \end{cases}, \begin{cases} y_1 = 1-\dfrac{3\sqrt{2}}{2} \\ y_2 = \dfrac{3\sqrt{2}}{2}+1 \end{cases},$$

Since the question requires $x-y$ to take the maximum value, the maximum value of $x-y$ when $x=\dfrac{3\sqrt{2}}{2}+2, y=1-\dfrac{3\sqrt{2}}{2}$ is $3\sqrt{2}+1$.

Ans: According to the question, it is to find the maximum value of $x-y$ under the constraints of the quadratic equation $x^2+y^2-4x-2y-4=0$, set up the corresponding conditional function and objective function, construct the Lagrangian function, solve the corresponding system of equations, and finally determine which set of values is the maximum value required in the question.

Lagrange number multiplication to solve the extreme value problem is a new way of thinking, this idea is different from method one and method two. Students need to determine the objective function and constraints in the topic, this method is different from the high school problem-solving ideas, students are no longer focusing on the equations and geometric graphs associated with the need to analyze the topic directly from the function perspective. This has a kind of impact on students' thinking, abandoning inertia thinking, pursuing the most original purpose of the topic, and exercising students' analytical ability.

ISSN 2616-5805 Vol. 5, Issue 3: 8-15, DOI: 10.25236/AJMS.2024.050302

4.2 Applications to elliptic problems

Conic curves are the second part of plane analytic geometry at the high school level, which students begin to learn after studying lines and circles. Students need to understand the background of conic curves, master the definition, standard equations, and simple properties of the ellipse, understand the definition, graphical properties, and standard equations of hyperbola and parabola, and further appreciate the idea of combining numbers and shapes[3]. Students learn ellipses based on straight lines and circles, and it is easy to connect straight lines and ellipses in their thinking. In addition to the topic of straight lines, geometric figures themselves also have many properties and test points, including distance and area is a common test points, but are also easy to combine with extreme value problems, the next two examples to illustrate.

Example 2: (2021. NSSE) Given that B is the upper vertex of the ellipse C : $\frac{x^2}{5}+y^2=1$ and the point P is on C , what is the maximum value of $|PB|$?

Solution 1: From the question, we have $a=\sqrt{5}$, $b=1$, so the upper vertex of the ellipse is $(0,1)$ .Set $P(x_1,y_1)$, $\frac{x_1^2}{5}+y_1^2=1$, $x_1^2+5y_1^2=5$,

$$|PB|=\sqrt{x_1^2+(y_1-1)^2}=\sqrt{-4y_1^2-2y_1+6}=\sqrt{-4(y_1+\frac{1}{4})^2+\frac{25}{4}}$$

Since $-1\le y_1\le 1$, the maximum value of $|PB|$ is $\frac{5}{2}$ when $y_1=-\frac{1}{4}$.

Ans: According to the equation of the ellipse in the question get the basic conditions to find the most value of the distance from a moving point on the ellipse to a fixed point. Students are prone to think of setting the coordinates of the point and listing the distance formula between the two points to find the maximum value. This method is a conventional method in high school, the need to bring the equation of the ellipse into the distance formula in the vertices of the formula, you can see that students need to have some arithmetic skills, the content of the quadratic function to pass, the process is complicated, the need to carry out a reasonable carry, and the instructions in a certain range of the maximum value. The following use of the condition of the extreme value to answer.

Solution 2: From the question, the upper vertex of the ellipse is $(0,1)$ Let $P=(x,y)$ be $f(x,y)=PB^2=x^2+(y-1)^2$, the Constraints $\varphi(x,y)=x^2+5y^2-5$ As a Lagrangian function $L(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y)$,

make the equation

$$\begin{cases} L_x=2x+2\lambda x=0 \\ L_y=2(y-1)+10\lambda y=0 \\ L_\lambda=x^2+5y^2-5=0 \end{cases}$$

Solving the system of equations gives $y=-\frac{1}{4}$, the $\begin{cases} x_1=-\frac{5\sqrt{3}}{4} \\ x_2=\frac{5\sqrt{3}}{4} \end{cases}$

Bringing the resulting solution to $f(x,y)$ yields the maximum value of $|PB|$ as $\frac{5}{2}$.

ISSN 2616-5805 Vol. 5, Issue 3: 8-15, DOI: 10.25236/AJMS.2024.050302

Ans: This question if the use of conditional extremes to construct the Lagrangian function is more convenient, the topic sets out the coordinates of the moving point after the objective function is obvious, the constraints are also relatively single, set out the system of equations to answer. Students who use this method need to consider from the perspective of the function and the derivative, after setting out the moving point indicates the distance, they no longer consider the equation of the ellipse with the association. The biggest obstacle for high school students in thinking in terms of utilizing Lagrange number multiplication is understanding partial derivatives. Initially studied in high school, the derivative, which in terms of advanced mathematics is the derivative of a unitary function, however, partial derivatives are in the context of the derivation of one of the unknowns in a multivariate function. Teachers need to pay attention to students' receptivity and comprehension when explaining to students, and students need to have certain arithmetic skills when solving systems of equations.

In addition to distance problems, area problems are also included in conic curves. Area problems are an important type of geometry problem in secondary school and are also high-frequency tests in analytic geometry, including the three tests of fixed value, range, and maximum value. To solve this type of problem, students need to find a reasonable relationship, construct an expression for the area, and then analyze it from the perspective of a function. The common core conditions for finding a reasonable relationship are given side length relationship, set position relationship, and given angle condition[8] . Students should pay attention to summarizing the methods and ideas of problem-solving in the daily learning process, and develop the ability to extract information on the topic. The following is an illustration of a problem about the extreme value of the area of an ellipse.

Example 3: Given that the equation of the ellipse C is $\frac{x^2}{4}+y^2=1$, where A is an upper vertex, B is a vertex, and the point P is on the ellipse, find the maximum value of the area of $\triangle ABP$?

Solution 1: The parametric equations of the ellipse from the question are $\begin{cases} x=2\cos\theta \\ y=\sin\theta \end{cases}$ $(0\le\theta\le 2\pi)$,and then $P(2\cos\theta,\sin\theta)$,

the equation of the line is $x+2y-2=0$,The distance from the point P to the line $d=\frac{|2\cos\theta+2\sin\theta-2|}{\sqrt{5}}$, the $|AB|=\sqrt{5}$ .the area of $\triangle ABP$ is

$$S=\frac{1}{2}|AB|\cdot d=\frac{1}{2}|2\cos\theta+2\sin\theta-2|=\frac{1}{2}|2\sqrt{2}\sin(\theta+\varphi)-2|$$

,When $\sin(\theta+\varphi)=-1$,The area of $\triangle ABP$ obtains the maximum value $\sqrt{2}+1$.

Ans: Algebraic, geometric, and parametric equations are commonly used in secondary schools to solve this type of area problem. The algebraic method is to set the coordinates of the moving point according to the equation of the ellipse, and then to find the height of the pair, and then solve the problem according to the formula for calculating the area, which requires students to have the corresponding problem-solving skills and solid arithmetic ability. The geometric method is to use the tangent line, and tangent point to get the corresponding distance, the distance of the most value is the most value of the area, this method requires students to have strong geometric thinking and intuitive imagination ability. Method one is to use the elliptic parametric equation to express the coordinates of the moving point, express the area, and then convert it to find the most value of the trigonometric function. This method is relatively simple, better for students to understand, easy to calculate, and needs to pay attention to the range of angles.

Solution 2: Let the coordinates of the point P be (x,y) , $\overrightarrow{AB}=(2,-1)$ By the definition of vector product, the area of $\triangle ABP$ is

$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AP}\right|=\frac{1}{2}\left\|\begin{matrix} i & j & k \\ 2 & -1 & 0 \\ x & y-1 & 0 \end{matrix}\right\|=\frac{1}{2}\left|0,0,x+2y-2\right|=\frac{1}{2}\left|x+2y-2\right|,$$

Constructing a Lagrangian function $L(x,y,\lambda)=(x+2y-2)^2+\lambda(1-\frac{x^2}{4}-y^2)$

Solving systems of equations $\begin{cases} L_x = 2(x+2y-2)-\frac{2\lambda x}{4}=0 \\ L_y = 4(x+2y-2)-2\lambda y = 0 \\ L_\lambda = 1-\frac{x^2}{4}-y^2=0 \end{cases}$

Solve for $\begin{cases} x_1=\sqrt{2} \\ x_2=-\sqrt{2} \end{cases}$, $\begin{cases} y_1=\frac{\sqrt{2}}{2} \\ y_2=-\frac{\sqrt{2}}{2} \end{cases}$, and

Then the maximum value of the area of $\triangle ABP$ is $\sqrt{2}+1$ when $x=-\sqrt{2}, y=-\frac{\sqrt{2}}{2}$.

Ans: Method 2 gives the idea of calculating the area of a triangle as a vector product, which also introduces the vector knowledge of higher mathematics into high school problem-solving. If students do not understand this method they can still use method 1 to list the expression of the area. By the objective function and constraints constitute the Lagrange function, solve the corresponding system of equations, and get the extreme value of the area. This method is more operational than seeking the maximum value of trigonometric functions, students are more likely to understand the principle, from the point of view of the solution, the method also requires students to have certain arithmetic skills.

5. Conclusion

The conic curve is the key content of high school mathematics, but also the core of plane analytic geometry, this part of the ellipse of the topic is more, relative to the hyperbola and parabola is more important. This is according to the general high school curriculum standards for plane analytic geometry content requirements. There are many types of area problems related to conic curves, and the methods of questioning are also various. In the process of arithmetic, students need to master the basic formulas, such as the chord length formula, the distance formula from the point to the straight line, the formula for the area of a triangle, and so on, and they also need to flexibly convert between algebra and geometry, and they even need to combine with the idea of function, which, to a certain extent, examines the students' ability of comprehensive application of the knowledge they have gained[9] .

Mathematics teaching at the high school level should not only teach students basic knowledge, but also transmit to students a mathematical idea, which is a kind of understanding of the nature of the mathematical sciences and the laws, and the essence of mathematical research[10]. As a secondary school mathematics teacher, it is more important to analyze and reflect on the teaching materials and the daily teaching process from a high point of view. Teachers should use the knowledge, ideas, and methods of higher mathematics to solve the problems in secondary school mathematics, study the problems from different perspectives, classify the existing mathematical fixed problems in high school, and gradually improve their knowledge system. For problem-solving, students should not aim to get the questions right, and teachers should not teach students to do the questions alone but should let students explore the nature of the questions in their daily learning, trace the roots, understand the source of knowledge, and experience the process of knowledge discovery. To achieve this purpose, teachers should lead students to understand the problem from a high point of view and solve the problem more broadly and conveniently. Students use the knowledge of higher mathematics can be subtle development of their

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creative ability, this ability will also be accompanied by students' lifelong development.

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