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# Solution of the Eigenfunction of Mixed Random Variables

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***Abstract:*** *The distribution function of mixed random variable can be split into a linear combination of a continuous distribution function and a discrete distribution function. In this paper, the expectation of mixed random variable is analyzed, and then the mathematical expectation of the function of mixed random variable is studied, and finally the general solution formula of the characteristic function of mixed random variable is obtained. The solution of the characteristic functions of the usual discrete and continuous random variables can also be analyzed by the ideas and methods in this paper, and relevant conclusion can also be drawn.*

***Keywords:*** *mixed random variables, eigenfunctions, distribution functions, stieltjes integrals*

## 1. Introduction

Eigenfunction is a powerful tool for dealing with probability theory problems, and the characteristic function is mainly divided into the eigenfunction of continuous random variables and the eigenfunction of discrete random variables in general textbooks. However, in practice, in addition to discrete and continuous types, there are also both non-discrete and non-continuous mixed random variables. How to solve the feature function of mixed random variables is an urgent problem to solve.

The probability density function is generally used for continuous random variables. The distribution columns are generally used for discrete random variables. Mixed random variables do not have distribution columns or density functions. Therefore, this paper plans to construct the feature function of mixed random variables from the perspective of distribution function.

For the study of mixed random variables, a lot of research has been done before. Wu Chuanju[1] constructed mixed random variables with the help of traditional discrete and continuous random variables in the article Examples and Analysis of Mixed Random Variables, and used the distribution function to reveal the value regularity of mixed random variables. He Xiaoxia[2] discussed the general solution of mathematical expectations of mixed random variables in the calculation and application of mixed random variables, and gave application examples in actuarial. Hu Xiaoshan[3] in the calculation of the numerical characteristics of mixed random variables by constructing the generalized inverse of the distribution function. The random variable is represented as a function of a uniformly distributed random variable, and then the expectation solving formula of the random variable function is used to solve the numerical characteristics of the mixed random variable. Ning Rongjian[4] constructed a distribution function on the basis of the law limit theory in the calculation of mathematical expectation and variance of mixed random variables based on distribution function, and used the distribution function to solve the mathematical expectation and variance of mixed random variables. Yao Yuanguo[5] discusses the calculation method of the distribution composed of two kinds of random variable functions, discrete and continuous, in the article Distribution of Mixed Functions of Two Different Types of Random Variables. Chen Hongyan[6] gave the calculation of several different types of random variable expectations in the calculation of random variable function expectation, and concluded the calculation of some mathematical expectations of mixed random variables. Xu Yongli[7] defines the probability distribution law of a mixed random variable in the article The Law of Probability Distribution of a Mixed Random Variable and Its Numerical Characteristics, and calculates the mathematical expectation and the mixed random variable given probability distribution column. He Xiaoxia[8] decomposed the general distribution function into continuous and discrete parts in using the distribution function decomposition to find the expectation of random variables. Then, according to the nature of the Stieltjes integral, the expectation of the mixed random variable is calculated.

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Relying on the previous research, this paper uses the distribution function to solve the characteristic function of mixed random variables. In this paper, the basis of probability limit theory of Lin Zhengyan [9], and the content of mathematical statistics and probability theory tutorial of Mao Shisong[10] are also referenced.

## 2. Preliminaries

**Definition 1**[10] Mixed random variable

A random variable is a real-valued function $U = U(\omega)$ defined on sample space $\Omega$. Discrete random variables have a finite number of values or can be listed. The value of a continuous random variable fills a certain interval $(a,b)$.Random variables that are neither discrete nor continuous are collectively referred to as mixed random variables.

**Definition 2**[10] Distribution function

The have random variable $U$, then distribution function $F(\mathrm{u}) = P(U \le u)\ (\mathrm{u} \in R)$.

When $U$ is discrete variable, the value of $U$ is $U_k(k=1,2,\cdots)$ and there is $(k=1,2,\cdots)$ $P(U=U_k)=p_k$, then $P(U=U_k)=p_k\ (k=1,2,\cdots)$ is distribution column, $F(U)=\sum_{k=1}^{+\infty} P_k$.

When $U$ is continuous variable, there is a $p(u)$ non-negative integrable function such that $F(u)=\int_{-\infty}^{u} p(t)\,dt$ is available for $\forall u \in R$, then $p(u)$ is density function.

**Definition 3**[9] Feature function

Random variable is $U$, distribution function is $F(u)$, $\varphi(t)=E\left(e^{itU}\right)=\int_{-\infty}^{+\infty} e^{itu}\,dF(u)$, $-\infty < t < +\infty$ is characteristic function, it is Fourier-Stieltjes transform.

When $U$ is a continuous, $\varphi(t)=E\left(e^{itU}\right)=\int_{-\infty}^{+\infty} e^{itu} p(u)\,du$, $-\infty < t < +\infty$.

When $U$ is a discrete, $\varphi(t)=E\left(e^{itU}\right)=\sum_{k=1}^{+\infty} e^{itu_k} p_k$, $-\infty < t < +\infty$, $p_k$, $k=1,2,\cdots$.

## 3. Main conclusions

### *3.1 The mathematical expectation of a mixed random variable*

In the classification of traditional random variables, random variables are generally divided into continuous and discrete, and $EU=\int_{-\infty}^{+\infty} u p(u)\,du$ is generally used to solve the mathematical expectations of continuous random variables; For the calculation discrete mathematical expectations, $EU=\sum_{i=1}^{\infty} U_i P_i$ is generally used to calculate them. In fact, random variables also have mixed random variables that are neither continuous nor discrete. For the calculation of the mathematical expectation of such random variables, because it does not have a density function and a distribution column, it can only rely on the distribution function to calculate its mathematical expectation.

**Lemma 1**[11] $F(u)$ is distribution function of $U$ and if $\int_{-\infty}^{+\infty} |u|\,dF(u) < \infty$, mathematical expectation is $EU=\int_{-\infty}^{+\infty} u\,dF(u)$.

The integral $\int_{-\infty}^{+\infty} u\,dF(u)$ in lemma 1 is the Rieman–Stieltjes integral.

In fact, mathematical expectation formula for discrete random variables and the mathematical expectation formula for continuous random variables are just special cases of the above equation integral.

In lemma 1, only the distribution function is involved, and mixed random variables have a

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distribution function, so lemma 1 can calculate the mathematical expectation of mixed random variables.

In fact, further analysis of lemma 1 yields the following other conclusions about the mathematical expectation solution of mixed random variables.

**Theorem 1**[12] $U$ is any random variable. If $EU$ is present, then there is $EU=\int_0^\infty p\left(U>v\right)dv-\int_0^\infty p\left(U\le -v\right)dv$.

**Proof** Distribution function is $F\left(u\right)=p\left(U\le u\right)$ of $U$, then there is $\int_0^\infty p\left(U>v\right)dv=\int_0^\infty\int_v^\infty dF\left(u\right)dv$

Using Fubini's theorem, transform the integral region, The original area is $D=\left\{\left(u,v\right)|\, v<u<\infty, 0<v<\infty\right\}$ ,The new region is $D_1=\left\{\left(u,v\right)|\, 0<v<u, 0<u<\infty\right\}$ , $D_1=D$.

Then

$$\int_0^\infty p\left(u>v\right)dy=\int_0^\infty\int_v^\infty dF\left(u\right)dv=\int_0^\infty\int_0^u dvdF\left(u\right)=\int_0^\infty udF\left(u\right)$$

Similarly, the have

$$\int_0^\infty p\left(u\le -v\right)dy=\int_0^\infty\int_{-\infty}^{-v} dF\left(u\right)dv=\int_{-\infty}^0\int_0^{-u} dvdF\left(u\right)=\int_{-\infty}^0 -udF\left(u\right).$$

$$\int_0^\infty p\left(U>v\right)dv-\int_0^\infty p\left(U\le -v\right)dy=\int_0^\infty udF\left(u\right)-\int_{-\infty}^0 -udF\left(u\right)=\int_{-\infty}^{+\infty} -udF\left(u\right)=EU.$$

Theorem 1 is also discussed in reference [12], but the literature discusses discrete or continuous cases, and this article further generalizes its conclusions to the case of mixed random variables.

In fact, in theorem 1, $v$ is an arbitrary number. The distribution function is $F_U\left(u\right)$ of $U$.

Theorem 1 can be written as

$$EU=\int_0^\infty\left[1-F_U\left(v\right)\right]dv-\int_0^\infty F_U\left(-v\right)dv=\int_0^\infty\left[1-F_U\left(v\right)-F_U\left(-v\right)\right]dv.$$

$$EU=\int_0^\infty\left[1-F_U\left(u\right)-F_U\left(-u\right)\right]dv.$$

In particular, when the random variable $U>0$ , the above equation can be reduced to $EU=\int_0^\infty\left[1-F_U\left(u\right)\right]du$.

When $U>0, U\in N^+, EU=\sum_{i=1}^{\infty}p\left(U\ge i\right)$,

When $U<0, EU=-\int_{-\infty}^0 F_U\left(u\right)du$,

When $-U>0, -U\in N^+, EU=-\sum_{i=1}^{\infty}p\left(U\le -i\right)$,

In the above discussion, the distribution function is analyzed as a whole, and in the text [13], The distribution function can be split into two parts, and this decomposition is unique.

**Lemma 2**[13] Distribution function $F\left(u\right)$ with a jump at point $a_i$ and a hop of $b_i$ , $\left(i=1,2,\cdots\right)$ ,then $F\left(u\right)=F_c\left(u\right)+F_d\left(u\right)$. $F_d\left(u\right)=\sum_{i=1}^{\infty}b_i\,|_{\left[d_i,\infty\right]}\left(u\right), F_c\left(u\right)=F\left(u\right)-F_d\left(u\right)$, where $F_c\left(u\right)$ is a continuous function.

In the above lemma, $F_c\left(u\right), F_d\left(u\right)$ is often not a distribution function, but may also be a distribution function. At most one of them can be zero of the $F_c\left(u\right), F_d\left(u\right)$. Of course, it makes no sense for $F_c\left(u\right), F_d\left(u\right)$ to be 0, in fact, when neither function is 0 (this is a mixed random variable), it can be regularized as a distribution function. It can have the following lemma.

**Lemma 3**[5] The distribution function $F\left(u\right)$ with a jump at point $a_i$ and a jump of $b_i$ ,

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$(i=1,2,\cdots)$ , then $F(u)=F_c(u)+F_d(u)$ , $F_d(u)=\sum_{i=1}^{\infty} b_i I_{[a_i,\infty]}(u)$ . $F_c(u)=F(u)-F_d(u)$ , $F_c(u)$ is continuous.

When $F_c(u), F_d(u)$ is not constant to 0, cause $\alpha=F_d(\infty)$,then

$F(u)=\alpha F_1(u)+(1-\alpha)F_2(u)$, thereinto $F_1(u)=\frac{1}{\alpha}F_d(u), F_2(u)=\frac{1}{1-\alpha}F_c(u)$.

Then $F_1(u)$ is the discrete and $F_2(u)$ is the continuous.

That is to say, the distribution function of any mixed variable can be split into the form of a linear combination of a discrete function and a continuous function.

From this, theorem 1 can be further analyzed, from the distribution function of the whole to solve the mathematical expectation to the disassembly into discrete and continuous two parts to solve the sum of mathematical expectations.

**Theorem 2** $U$ is a mixed variable, $F(u)$ has at most a hop break point $\{u_i, i=1,2,\cdots\}$ , $F(u)=\alpha F_1(u)+(1-\alpha)F_2(u)$ , $0<\alpha<1$ , $F_1(u)$ is a discrete function, and $F_2(u)$ is a continuous function, then $EU=\alpha\sum_{i=1}^{\infty} u_i p_i+(1-\alpha)\int_{-\infty}^{+\infty} up(u)du$ .

**Proof**

$$EU=\int_{-\infty}^{+\infty} udF(u), F(u)=\alpha F_1(u)+(1-\alpha)F_2(u).$$

$$\begin{aligned} EU=\int_{-\infty}^{+\infty} udF(u) &= \int_{-\infty}^{+\infty} udF\left[\alpha F_1(u)+(1-\alpha)F_2(u)\right] \\ &= \alpha\int_{-\infty}^{+\infty} udF_1(u)+(1-\alpha)\int_{-\infty}^{+\infty} udF_2(u). \end{aligned}$$

$p(u)$ is the density function of $F_2(u)$. $p_i$ is the distribution column of $F_1(u)$.

The above equations can be reduced to $EU=\alpha\sum_{i=1}^{\infty} x_i p_i+(1-\alpha)\int_{-\infty}^{+\infty} up(u)du$ . The operational properties of the Stieltjes integral used in the above integrals.

### *3.2 The mathematical expectation of function*

$EU$ is only the simplest origin moment, how to solve the other origin moment $EU^k$, if $EU^k$ can be solved, in moment theory, the central moment $E(U-EU)^k$ can be expressed as a function of the origin moment, that is, the $k$ order origin moment can be solved, then there is an $k$ order center moment can also be solved.

In fact, if there is a mixed variable $V=g(U)$, if the $EV=E(g(U))$ can be calculated, the above problems can be solved, and even $E(e^{itu})$ can be used to find the characteristic function of the mixed variable, and then analyze the mathematical expectation of $V=g(U)$. The $g(U)$ here is function that may map the mixed variable $U$ to the mixed random $V$ .

If $U \to V$ is a mixed type and the map is ordinary (discrete or continuous), or the principle is the same for the ordinary type to be mixed.

**Theorem 3** $U$ is any variable, $V=g(U)$ is function, then

$$EV=Eg(U)=\int_{-\infty}^{\infty} g(u)dF_U(u).$$

**Proof** that theorem 1 has arbitrary random variables $X$, all with

$$EU=\int_0^{\infty}\left[1-F_U(u)-F_U(-u)\right]dU,\ V=g(U), F_V(v)=p(V\le v)=p(g(U)\le v)$$

Then

$$EU=E(g(u))=\int_0^{\infty}\left[1-F_V(v)-F_V(-v)\right]dv=\int_0^{\infty}\left[p(V\ge v)-p(V<-v)\right]dv$$

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$$=\int_0^{\infty} p\left(g\left(U\right)\geq v\right)dv-\int_0^{\infty} p\left(g\left(u\right)<-v\right)dv$$
$$=\int_0^{\infty}\int_{g(u)\geq v} dF_U\left(u\right)dv-\int_0^{\infty}\int_{g(u)<-v} dF_U\left(u\right)dv.$$

Integral region transformation, integral region transformation for Stieltjes integrals

$$D_1=\left\{\left(u,v\right)\mid g\left(u\right)\geq v, 0<v<+\infty\right\}\Rightarrow D_1'=\left\{\left(u,v\right)\mid 0<v\leq g\left(u\right), g\left(u\right)>0\right\}$$
$$D_2=\left\{\left(u,v\right)\mid g\left(u\right)<-v, 0<v<+\infty\right\}\Rightarrow D_2'=\left\{\left(u,v\right)\mid 0<v<-g\left(u\right), g\left(u\right)<0\right\}$$

Then

$$EV=\int_0^{\infty}\int_{g(u)\geq v} dF_U\left(u\right)dv-\int_0^{\infty}\int_{g(U)<-v} dF_U\left(u\right)dv$$
$$=\int_{g(u)>0}\int_0^{g(u)} dv dF_U\left(u\right)-\int_{g(U)<0}\int_0^{-g(U)} dv dF_U\left(u\right)$$
$$=\int_{g(U)>0} g\left(U\right)dF_U\left(u\right)+\int_{g(U)<0} g\left(U\right)dF_U\left(u\right)$$
$$=\int_{g(u)\in R} g\left(U\right)dF_U\left(u\right)=\int_{-\infty}^{+\infty} g\left(U\right)dF_U\left(u\right)$$

Proof complete.

The integral here is the Stieltjes integral, not the equivalence integral, the formula here applies to any random variables. In fact, when $U$ is continuous, $dF_U\left(u\right)=p\left(u\right)$,

There is $EV=Eg\left(U\right)=\int_{-\infty}^{+\infty} g\left(U\right)p\left(U\right)dU$ .

When $U$ is discrete, $F_U\left(u\right)=\sum_{i=1}^{\infty} g\left(u_i\right)p_i$ , there is

$$EV=Eg\left(U\right)=\sum_{i=1}^{\infty} g\left(U_i\right)p_i .$$

When $U$ is mixed, the distribution function $F_U\left(x\right)$ can be split into the sum of the two distribution functions $F_1(u)$ and $F_2(u)$. That is, $F_U\left(u\right)=\alpha F_1\left(u\right)+\left(1-\alpha\right)F_2\left(u\right)$ , $0<\alpha<1$ . $F_1\left(u\right)$ is the discrete distribution function, and $F_2\left(u\right)$ is the continuous distribution function.

$$F_1\left(u\right)=\sum_{j=1}^{\infty} p_{1j} , dF_2\left(u\right)=p_2\left(u\right).$$

Then theorem 3 can be rewritten as

$$EV=Eg\left(U\right)=\int_{-\infty}^{+\infty} g\left(U\right)dF_U\left(u\right)=\int_{-\infty}^{+\infty} g\left(U\right)d\left[\alpha F_1\left(u\right)+\left(1-\alpha\right)F_2\left(u\right)\right]$$
$$=\alpha\int_{-\infty}^{+\infty} g\left(U\right)dF_1\left(u\right)+\left(1-\alpha\right)\int_{-\infty}^{+\infty} g\left(U\right)dF_2\left(u\right)$$
$$=\alpha\sum_{j=1}^{\infty} g\left(U_j\right)p_{1j}+\left(1-\alpha\right)\int_{-\infty}^{+\infty} g\left(U\right)p_2\left(u\right)du .$$

### *3.3 Solution of the feature function of mixed random variables*

In preliminary knowledge section, eigenfunction $\varphi\left(t\right)=E\left(e^{itu}\right)=\int_{-\infty}^{+\infty} e^{itu}dF\left(u\right)$ , and eigenfunction is the Fourier-Stieltjes transform. Combined with above theorem, the eigenfunction $\varphi\left(t\right)$ can be further analyzed and studied. $V=g\left(U\right)=e^{itU}$ .

**Theorem 4** $U$ is mixed random variable, $F\left(u\right)$ is distribution function, $\varphi(t)$ is characteristic function, then the $F_U\left(u\right)=\alpha F_1\left(u\right)+\left(1-\alpha\right)F_2\left(u\right), 0<\alpha<1$ . $F_1\left(u\right)$ is the discrete distribution function and $F_2\left(u\right)$ is the continuous function.

$$\varphi\left(t\right)=\alpha\sum_{k=1}^{\infty} e^{itu_k}p_{1j}+\left(1-\alpha\right)\int_{-\infty}^{+\infty} e^{itu}p_2\left(u\right)du .$$

**Proof**

$$\varphi(t) = E\left(e^{itu}\right) = \int_{-\infty}^{+\infty} e^{itu} dF_U(u)$$
$$= \int_{-\infty}^{+\infty} e^{itU} d\left[\alpha F_1(u) + (1-\alpha) F_2(u)\right]$$
$$= \alpha \int_{-\infty}^{+\infty} e^{itU} dF_1(u) + (1-\alpha) \int_{-\infty}^{+\infty} e^{itU} dF_2(u)$$
$$= \alpha \sum_{j=1}^{\infty} e^{itU} p_{1k} + (1-\alpha) \int_{-\infty}^{+\infty} e^{itU} p_2(u) du$$

Proof complete.

The mixed random variable eigenfunction is a linear combination of discrete good continuous eigenfunctions. In the past, the problem of mixed random variables was often solved only by using the distribution function. Here, the characteristic function of a mixed random variable is obtained. Then, problem of analyzing number of features or probability of mixed random variables can also be solved by the feature function.

In fact, in the study of continuous or discrete variables, distribution function can be solved by the eigenfunction through the reversal formula, and there is also such a conclusion in the mixed random variable here, which can also reflect a deeper unique theorem, that is, There is a unique correspondence between the distribution function and the characteristic function. Many conclusions about the characteristic functions of discrete and continuous can often be generalized to case of mixed variables.

## 4. Conclusions

Starting from the analysis of the mathematical expectation of mixed random variables, this paper further analyzes the mathematical expectation of mixed random variable functions, and then draws a general conclusion about the characteristic functions of mixed random variables.

$$\varphi(t) = \alpha \sum_{j=1}^{\infty} e^{itU} p_{1k} + (1-\alpha) \int_{-\infty}^{+\infty} e^{itU} p_2(u) du$$

In the process of solving the characteristic function, some general formulas for solving the mathematical expectation, mathematical expectations functions are obtained. It is of reference for further discussion of random variable variance and high-order moment, and also analyzes many conclusions about the expectations ordinary variables, which can also be derived from the formulas involved in this paper. Mixed random variables are closely combined with ordinary variables, many conclusions have similarities, and the characteristic functions of mixed random variables also provide another way to study mixed random variables in addition to the distribution function.

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## References

*[1] WU Chuanju, WU Lei, XU Mengbin, YANG Rui. Examples and analysis of mixed random variables [J]. Journal of Science and Technology, 2014, 4(34):31-33.*

*[2] HE Xiaoxia, HOU Xuan, LI Chunli. Calculation and application of mixed random variable expectation [J]. University Mathematics,2014,1(30):101-103.*

*[3] HU Xiaoshan, LIU Jicheng. Calculation of numerical characteristics of mixed random variables [J]. University Mathematics, 2016, 2(32):86-90.*

*[4] NING Rongjian, YU Bingsen. Calculation of mathematical expectation and variance of mixed random variables based on distribution function [J]. University Mathematics,2015,2(31):48-52.*

*[5] Yao Yuanguo. Distribution of mixed functions of two different types of random variables[J]. Journal of Bose Institute,2007,3(20):55-57.*

*[6] CHEN Hongyan, DENG Zhen. Calculation of mathematical expectation of random variables[J]. Journal of Natural Science of Harbin Normal University,2015,5(31),44-45.*

*[7] Xu Yongli. Probability distribution law and numerical characteristics of a mixed random variable[J]. Advanced Mathematics Research, 2015, 4(18):125-126.*

ISSN 2616-5775 Vol. 6, Issue 3: 28-34, DOI: 10.25236/AJCIS.2023.060304

*[8] HE Xiaoxia. Using distribution function decomposition to find mathematical expectation of random variables[J]. Journal of Advanced Mathematics, 2015, 4(8):60-62.*
*[9] LIN Zhengyan, LU Chuanrong, SU Zhonggen. Fundamentals of probability limit theory[M]. Higher Education Press, 1999.*
*[10] MAO Shisong, CHENG Yiming, PU Xiaolong. Probability Theory and Mathematical Statistics (Second Edition) [M]. Higher Education Press, 2011.*
*[11] Liang Zhishun. eds. Probability theory and mathematical statistics[M]. Higher Education Press, 1998.*
*[12] Cai Zelin. A concise proof of the mathematical expectation formula for the function of random variables[J]. journal of hubei normal university,1999,19(4):86-87.*
*[13] Kai Lai Chung. A course in probability theory [M]. California: Academic press,2001.*