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Variants of the Pythagorean Theorem

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ABSTRACT: *In this project, we look at variants of the Pythagorean theorem. The Pythagorean theorem says that in a in a right angled triangle, $a^2+b^2=c^2$. In this project, we will use the Pythagorean theorem in the two dimensional rectangular coordinate system and The three dimensional coordinates system to solve specific mathematic problem.*

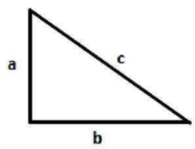
KEY WORDS: *Pythagorean Theorems, stereographic projection, modular arithmetic*

1. Background of the Pythagorean Theorem

1.1 History

In China, the first person who put forward the Pythagorean theorem was Shang Gao from the Zhou dynasty who come up with the theory of "three Hooks, four shares sad five strings"[1]. In the West, the first person who put forward and prove this theorem was Pythagorean in ancient Greece in the 6th century BC, who proved by deductive method that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the two sides[2][3].

1.2 Summary



In a right angled triangle, the longest side of the triangle with is also corresponding to the right angle (the 90 degree angle) is the side which will be named ‘c’. In the Pythagorean theorem, the two right-angle sides which are named ‘a’ and ‘b’ are not important, they can either be called ’a’ or ‘b’, but the side ‘c’ is always the longest side that correspond to the right angle.

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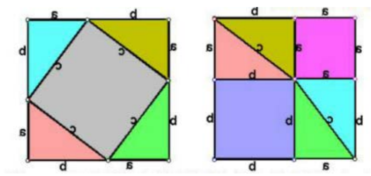
Consider that there are 8 congruent triangles and they are divided into two group. Let the shorter right-angle side equals to ‘a’ and the longer right-angle side equals to ‘b’. The first 4 congruent triangles make up a square that all of its 4 sides equals to ‘a + b’. The second 4 congruent triangles also make up a square that its 4 sides also equals to ‘a + b’, but they were formed in different ways. Most importantly, since both of the two squares have the same side length which is ‘a + b’ so they have the same area.

$$a\times a + b\times b + 4\times a\times b\times\frac{1}{2} = 4\times a\times b\times\frac{1}{2} + c\times c$$

$$a^2 + b^2 + 2ab = 2ab + c^2$$

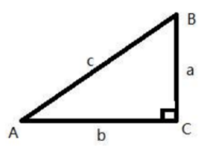
$$a^2 + b^2 = c^2$$



2.Primitive Triples

Definition: a primitive integers triple is a set of 3 integers with could be written as a form that $a^2 + b^2 = c^2$, which integer a and integer b are the two smaller integers among the three integers. Most importantly, there does not exists an integer d that could exact divide both a b c except 1.

We could also link primitive integers triple with the Pythagorean theorem. We can use a right angled triangle to explain this concept. In a right angled triangle we could proof that the square of the sum of the two right-angled sides equals to the square of the hypotenuse. Primitive integers triple is a special condition in the Pythagorean theorem which the three integers have no other common factor except 1.



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Example:

1. Integers 3, 4, 5 forms a primitive integer triple because $3^2 + 4^2 = 5^2$

2. Integers 5, 12, 13 forms a primitive integer triple because $5^2 + 12^2 = 13^2$

We could notice from these two examples that in the first example, the three numbers could not be divided by another integer except . In example 2 we could notice that 5, 12 ,13 do not have a common factor except 1.

Lemma there are infinite primitive integers that could satisfied $a^2 + b^2 = c^2$.

Proof. Divide both side of the equation by c^2. Let $\frac{a^2}{c^2}$ equals to x^2 Let $\frac{b^2}{c^2}$ equals to y^2,

then the equation equals to $x^2 + y^2 = 1$.

Select the point A (0,1) on the curve. Let B (p, 0) be a point on the x axis which p is a rational number. Connect point A and Point B. The line that connects point AB equals to y= $-\frac{1}{p}$x +1. Substitute y=$-\frac{1}{p}x + 1$ into $x^2 + y^2 = 1$, then $x^2 + (-\frac{1}{p}x + 1)^2 = 1$.So $(\frac{1}{p^2}+1)x^2 + -\frac{2}{p}x = 0$.

According to Vieta's formulas, the solution except 0 equals to $\frac{2p}{1+p^2}$. Since it is also on the line, so the other coordinate equals to $\frac{p^2-1}{p^2+1}$.

As a result, all numbers that satisfy the condition that $(\frac{2p}{1+p^2}, \frac{p^2-1}{p^2+1})$ could be a solution for the equation, as a result, there are infinite solutions.

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3.Derived problems

3.1 There are infinite primitive integers quadruples (a, b, c, d) satisfying $a^2 + b^2 + c^2 = d^2$.

Proof. Since $a^2 + b^2 + c^2 = d^2$

Divide both side by d^2 then the equation equals $\frac{a^2}{d^2}+\frac{b^2}{d^2}+\frac{c^2}{d^2} = \frac{d^2}{d^2}$

Let $\frac{a^2}{d^2} = x^2$, $\frac{b^2}{d^2}=y^2$, $\frac{c^2}{d^2}=z^2$, so $x^2+y^2+z^2=1$.We can say that the

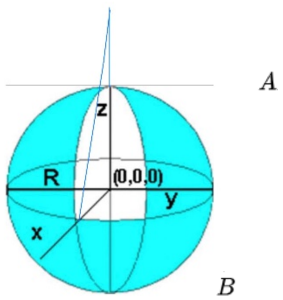

image of the equation $x^2+y^2+z^2=1$ is a sphere in a three

dimensional coordinate system.

Let point A = (1, 0, 0). Since $1^2+0^2+0^2=1$, point A is on the equation $x^2+y^2+z^2=1$.

Let point B = (p, q, 0),which is on the xy-plane. Variables p and q are rational.By connecting point A with point P, we get a line AB. It can be written as

$$\begin{cases} x = pt \\ y = qt \quad ,t \in R \\ z = 1 - t \end{cases}$$

Substitute this into the equation $x^2 + y^2 + z^2 = 1$. We have

$$(pt)^2 + (qt)^2 + (1 - t)^2 = 1$$

Solve this equation. We obtain the other intersection point, whose coordinate is

$$x = \frac{2p}{p^2 + q^2 + 1}$$

$$y = \frac{2q}{p^2 + q^2 + 1}$$

$$z = \frac{p^2 + q^2 - 1}{p^2 + q^2 + 1}$$

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Since p and q are both rational numbers, and there are infinite rational numbers, so the number of rational solution for $a^2 + b^2 + c^2 = d^2$. are supposed to be infinite.

Module

Definition: A fundamental algebraic structures used in abstract algebra. If "a ÷ b=c……d", we could say that a= b mod d. In the equation "a ÷ b=c……d" all of the number a, b, c, d should be integers, In module, the answer c is not important, since it is always written as 'remainder' mod 'devisor'. Most importantly, all numbers divided by b that has a remainder d could be written as 'd mod b'.

Example: 1. 5÷3=1……2 .In this case, the divisor is "3" and the remainder is "2", so it could be written as "2 mod 3".

2. 23÷3=7……2. In this case, the divisor is "3" and the remainder is "2", so it could be written as "2 mod 3"

As a result, we can see from this 2 examples that even though this two equations have different dividend and different answer. However, both of them could be written as "2 mod 3". We know that module is only related to the divisor and the remainder.

3.2. $x^2 + y^2 = 3$ has no rational solution.

Proof. Assume that $x^2 + y^2 = 3$ has rational solution. Let $x=(\frac{a}{b})$ $y=(\frac{d}{c})$,where $\frac{a}{b}$ and $\frac{d}{c}$ are irreducible Since $(\frac{a}{b})^2+(\frac{d}{c})^2=3$.

$a^2c^2 + d^2b^2 = 3b^2c^2$

Since $3b^2c^2 \equiv 0 \bmod 3$, $a^2c^2 + d^2b^2 \equiv 0 \bmod 3$. There are 2 conditions :

1. Both a^2c^2 and $b^2d^2 \equiv 0 \bmod 3$

2. a^2c^2 equals mod 3 and $d^2b^2 \equiv 2 \bmod 3$ or $a^2c^2 \equiv 2 \bmod 3$ and d^2b^2 equals 1 mod 3

However, square numbers only $\equiv$ 1 mod 3 or 0 mod 3, so only condition 1 is possible.

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Both a and d $\equiv$ 0 mod 3. So a^2c^2 and b^2d^2 $\equiv$ 0 mod 9, but $3b^2c^2$ could not be written as 0 mod 9 since neither b or c could be written as 0 mod 3 or it would contradict that $\frac{a}{b}$ and $\frac{d}{c}$ are irreducible. So there exists no rational solution for $x^2 + y^2 = 3$.

3.3 If there exists a rational solution of the equation $Ax^2 + By^2 = 1$ for A, B being non-zero numbers, then there exists infinitely many.

Proof. The image of $Ax^2 + By^2 = 1$ is a conic in the rectangular coordinate system. Assume that point P with coordinates (a, b) is a point on the function $Ax^2 + By^2 = 1$, both c and d are rational number. Point Q is a point on the x-axis with coordinate (c, 0). Connect point A with point P, assume the function that connect the two points equal to y.

Since it is a two dimensional rectangular coordinate and the line that connects the point is a straight line, so y=kx +t. Since the coordinates of point Q is (c, 0) and point P is (a,b).

It is easy to reckon that

$k = \frac{b}{a-c}$, $t = \frac{-bc}{a-c}$ and $y = \frac{b}{a-c}x - \frac{bc}{a-c}$

Substitute the line y into $Ax^2 + By^2 = 1$ to get the equation

$Ax^2 + B\times(\frac{b}{a-c}\times x - \frac{bc}{a-c})^2 = 1$

Since the sum of the solution of the equation equals to $\frac{(-1)\times(the\ coefficient\ of\ x)}{the\ coefficient\ of\ x^2}$, the sum of the solution = $\frac{-2Bkt}{A+Bk^2}$. As it had mentioned in the start, point A is a point of the function, so the other solution equals to $\frac{-2Bkt}{A+Bk^2}$ - a. Substitute $x = \frac{-2Bkt}{A+Bk^2} - a$ into line PQ. We have $y = -\frac{2Bbkt}{(a-c)(A+Bk^2)} - \frac{b(a+c)}{a-c}$.

So the coordinate equals to $(\frac{-2Bkt}{A+Bk^2} - a, -\frac{2Bbkt}{(a-c)(A+Bk^2)} - \frac{b(a+c)}{a-c})$.

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Since a, b, c, k, t, A, B are all rational numbers and there exists infinity rational numbers, there exists infinitely many rational solutions.

3.4 If $d \equiv 3(mod\ 4)$, then $x^2+y^2=d$ have no integer solution.

Proof. Since d ≡3 mod 4, so $x^2+y^2 \equiv 3$ mod 4. The remainder of A square number divided by 4, could only be 1 or 0.

As a result, $x^2 \equiv 0\ or\ 1(\ mod\ 4)$ **.** there exists 2 conditions.

$$x^2 \equiv\ 1\ mod\ 4$$

$$x^2 \equiv\ 0\ mod\ 4$$

In condition 1, if $x^2 \equiv\ 1\ (mod\ 4)$, in order to make $x^2+y^2 \equiv\ 3\ (mod\ 4)$, $y^2 \equiv\ 2\ (mod\ 4)$. However, y^2 is also a square number, which it could only $\equiv\ 0\ or\ 1\ (mod\ 4)$. As a result, condition 1 is not possible.

In condition 2, if $x^2 \equiv 0\ (mod\ 4)$, in order to make $x^2+y^2 \equiv 3(mod\ 4), y^2 \equiv 3(mod\ 4)$. However, y^2 is also a square number, which it could only equals to $0\ or\ 1\ (mod\ 4)$. As a result condition 2 is not possible.

Since both of the two conditions are not possible, the statement is proved.

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