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Polyharmonic Fundamental Solutions in Lipschitz Graph Domain in $\mathbb{R}^2$

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ABSTRACT. *In this article, we study polyharmonic fundamental solutions in Lipschitz graph domain in* $\mathbb{R}^2$*. By the ultraspherical polynomials and the definition of the polyharmonic fundamental solutions, we give a harmonic recursion formulate of the polyharmonic fundamental solutions.*

KEYWORDS: *Polyharmonic fundamental solutions, ultraspherical polynomials, Lipschitz graph domain*

1. Introduction

In recent years, partial differential equations has been a very important study. A lot of research has been done on the boundary value problem of higher order partial differential equations. In [2], Du introduced higher order conjugate Poisson and Poisson kernels and polyharmonic fundamental solutions. By applied these, Du solved three classes of boundary value problems in Lipschitz domains in $\mathbb{R}^n\left(n\geq 3\right)$. In [3-7], Du studied Dirichlet problems with L^p boundary data for polyharmonic functions in different regular domains (such as, the unit disc, the upper-half plane, the unit ball and the upper-half plane). By constructing higher order conjugate Poisson and Poisson kernels of different regular regions, Du gived the integral representation solutions of the problem on the corresponding domains. We can find that the key to solve Dirichlet boundary value problems is constructing higher order conjugate Poisson and Poisson kernels. Similary, if we want to study polyharmonic Neumann problem in Lipschitz graph domain in $\mathbb{R}^2$, we need to construct polyharmonic fundamental solutions (which are higher order analogues of the calssical fundamental solution of the Laplacian). In this paper, we study polyharmonic fundamental solutions in Lipschitz graph domain in $\mathbb{R}^2$.

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2. Some definitions and known results

In this section, we recall some known facts including the definition of Lipschitz graph domain and some knowledge about the ultraspherical polynomials.

Definition 2.1 Let

$$D=\{(x_1,x_2)\in\mathbb{R}^2 : x_2>\varphi(x_1), x_1\in\mathbb{R}\} \tag{2.1}$$

Where $\varphi:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous; namely, $\left|\varphi(x_1)-\varphi(x_1^{'})\right|\le L\left|x_1-x_1^{'}\right|$ with $0<L<\infty$. Then D is a Lipschitz graph domain in $\mathbb{R}^2$.

In what follows, we introduce ultraspherical polynomails [1], [8], $P_l^{(\lambda)}$ and $Q_l^{(\lambda)}$, which can be respectively defined by the generating functions

$$(1-2r\xi+r^2)^{-\lambda}=\sum_{l=0}^{\infty}P_l^{(\lambda)}(\xi)r^l \tag{2.2}$$

And

$$(1-2r\xi+r^2)^{-\lambda}\log(1-2r\xi+r^2)=\sum_{l=0}^{\infty}Q_l^{(\lambda)}(\xi)r^l \tag{2.3}$$

Where $\lambda\neq 0$, $0\le|r|<1$ and $|\xi|\le 1$. $P_l^{(\lambda)}$ and $Q_l^{(\lambda)}$ have the following explicit expressions:

$$\begin{aligned}P_l^{(\lambda)}(\xi)&=\frac{1}{l!}\{\frac{d^l}{dr^l}[(1-2r\xi+r^2)^{-\lambda}]\}_{r=0}\\&=\sum_{j=0}^{\left[\frac{l}{2}\right]}(-1)^j\frac{\Gamma(l-j+\lambda)}{\Gamma(\lambda)j!(l-2j)!}(2\xi)^{l-2j}\end{aligned} \tag{2.4}$$

And

$$\begin{aligned}Q_l^{(\lambda)}(\xi)&=-\frac{d}{d\lambda}\left[P_l^{(\lambda)}(\xi)\right]\\&=\sum_{j=0}^{\left[\frac{l}{2}\right]}\sum_{k=0}^{l-j-1}(-1)^{j+1}\frac{\Gamma(l-j+\lambda)}{(\lambda+k)\Gamma(\lambda)j!(l-2j)!}(2\xi)^{l-2j}\end{aligned} \tag{2.5}$$

Where $\left[\frac{l}{2}\right]$ denotes the integer of $\frac{l}{2}$.

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Definition **2.2.** Let D be a Lipschitz graph domain in $\mathbb{R}^2$ with Lipschitz graph boundary ∂D. For sufficiently large $|\zeta|$, $m \geq 2$, let f be a continuous function defined in $\mathbb{R}^2$ that can be expanded as

$$f(\zeta) = S.P.[f](\zeta) + I.P.[f](\zeta) \tag{2.6}$$

Where the coefficient functions $c_k(\zeta)$ are continuous in $\mathbb{R}^2$. At the same time, $S.P.[f](\zeta) = \sum_{k=-1}^{m} c_k(\zeta)|\zeta|^k$ and $I.P.[f](\zeta) = \sum_{k=2}^{\infty} c_{-k}(\zeta)|\zeta|^{-k}$. For sufficiently large R, B_R deontes a ball centered at the origin with radius R. If $I.P.[f]$ is L^p integrable on $\partial D \setminus B_R$, then $S.P.[f]$ is called the singular part of f and $I.P.[f]$ is called the integral part of f in the sense of L^p integrable , $p > 1$.

3. Polyharmonic fundamental solutions

In [2], Du introduced the polyharmonic fundamental solution in $\mathbb{R}^n (n \geq 3)$, which are higher order analogues of the classical fundamental solution. By the same way, we can give the polyharmonic fundamental solutions in Lipschitz graph domain in $\mathbb{R}^2$.

Definition **3.1.** Suppose that $X = (x_1, x_2)$, $V = (v_1, v_2) \in \mathbb{R}^2$ and $X \neq V$. Let

$$\mathcal{D}_m(X,V) = \frac{1}{2\pi} \cdot \frac{1}{[2(m-1)!!]^2} |X - V|^{2(m-1)} [\log|X - V| + \frac{1}{2} - \sum_{k=1}^{m-1} \frac{1}{k}] \tag{3.1}$$

Then

$$\Delta \mathcal{D}_1(X,V) = 0 \text{ and } \Delta \mathcal{D}_m(X,V) = \mathcal{D}_{m-1}(X,V),\ m \geq 2. \tag{3.2}$$

Suppose D is a Lipschitz graph domain in $\mathbb{R}^2$ with Lipschitz graph boundary

∂D. For any $X = (x_1, x_2)$, $V = (v_1, v_2) \in \overline{D}$ and $X \neq V$, it is easy to find that $\mathcal{D}_m(X,V)$ is not L^p integrable when the other variable is fixed, where $p > 1$. We must abandon singular part of $\mathcal{D}_m(X,V)$ in order to ensure the L^p integrability of $\mathcal{D}_m(X,V)$.

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Lemma **3.2.** Let D be a Lipschitz graph domain in $\mathbb{R}^2$ with Lipschitz graph Boundary ∂D. Suppose that $X=(x_1,x_2)$, $V=(v_1,v_2)\in\bar{D}$ and $X\neq V$. Let

$$\mathcal{K}_m(X,V)=\mathcal{D}_m(X,V)-S.P.[\mathcal{D}_m](X,V) \tag{3.3}$$

Where

$$S.P.[\mathcal{D}_m](X,V)=\frac{1}{2\pi}\cdot\frac{1}{[2(\text{m-1})!!]^2}\{\frac{1}{2}\sum_{l=0}^{2m-1}Q_l^{(1-m)}(X_{s^1}\cdot \mathrm{V}_{s^1})\times\max(|X|^{2(m-1)},|V|^{2(m-1)})$$
$$\times\min(\left(\frac{|X|}{|V|}\right)^l,\left(\frac{|X|}{|V|}\right)^{-l})+\sum_{l=0}^{2m-1}P_l^{(1-m)}(X_{s^1}\cdot \mathrm{V}_{s^1})\times\min(\left(\frac{|X|}{|V|}\right)^l,\left(\frac{|X|}{|V|}\right)^{-l})$$
$$\times\max(|X|^{2(m-1)},|V|^{2(m-1)})\times[\log\max(|X|,|V|)+\frac{1}{2}-\sum_{k=1}^{m-1}\frac{1}{k}]\}$$

for any $m\in\mathbb{N}$ and $m\geq 2$. Then $\mathcal{K}_m$ is called the m th order polyharmonic fundamental solution for any $m\in\mathbb{N}$.

In what follows, we will introduce some properties of the polyharmonic fundamental solution in Lipschitz graph domain in $\mathbb{R}^2$. From the definition of $\mathcal{K}_m$, we can easy to get that $\mathcal{K}_m$ is symmetric.More precisely,

Proposition **3.3.** Let D be a Lipschitz graph domain in $\mathbb{R}^2$ with Lipschitz graph boundary ∂D, and $\mathcal{K}_m$ be the m th order polyharmonic fundamental solution. Suppose that $X=(x_1,x_2)$, $V=(v_1,v_2)\in\bar{D}$ and $X\neq V$. Then

$$\mathcal{K}_m(X,V)=\mathcal{K}_m(V,X) \tag{3.4}$$

Theorem **3.4.** Let $\{\mathcal{K}_m\}_{m=1}^{\infty}$ be the sequence of the polyharmonic fundamental solutions, and D be a Lipschitz graph domain in $\mathbb{R}^2$ with Lipschitz graph boundary ∂D. Then for any $(X,Q)\in D_c\times\{Q\in\partial D:|Q|>T\}$, and $m\geq 2$,

$$|\mathcal{K}_m(X,Q)|\leq M\frac{1}{\left(1+|Q|^2\right)^{\frac{1+\varepsilon}{2}}} \tag{3.5}$$

Where $0<\varepsilon<1$, D_c is any compact subset of $\bar{D}$, T is a sufficiently large real number and M is a constant depending only on ε, D_c and T.

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Proof: By the definition and Taylor's expansion, for sufficiently large $|V|>|X|$,

$$I.P.[\mathcal{D}_m](X,V)=A_m\left[C_m(X,V)+\tilde{C}_m(X,V)\log|V|\right]\frac{1}{|V|^2} \qquad (3.6)$$

Where A_m is a constant depending only on m,

$$\begin{aligned}C_m(X,V)=|X|^{2m}\{\frac{d^{2m}}{dr^{2m}}[&\left(1-2r\left(X_{S^1}\cdot V_{S^1}\right)+r^2\right)^{m-1}] \\ &\times\left[\frac{1}{2}\log\left(1-2r\left(X_{S^1}\cdot V_{S^1}\right)+r^2\right)+\frac{1}{2}-\sum_{k=1}^{m-1}\frac{1}{k}\right]]\}_{r=\vartheta}\end{aligned} \qquad (3.7)$$

And

$$\tilde{C}_m(X,V)=|X|^{2m}\left\{\frac{d^{2m}}{dr^{2m}}\left[\left(1-2r\left(X_{S^1}\cdot V_{S^1}\right)+r^2\right)^{m-1}\right]\right\}_{r=\rho} \qquad (3.8)$$

With $0<\vartheta,\rho<\frac{|X|}{|V|}<1$. We note that $\lim_{|V|\to\infty}\frac{\log|V|}{|V|^{\varepsilon}}=0$, for any $\varepsilon>0$. Then for any

$m\geq 2$, we have

$$\left|\mathcal{K}_m(X,Q)\right|\leq M\frac{1}{\left(1+|Q|^2\right)^{\frac{1+\varepsilon}{2}}}$$

For any $(X,Q)\in D_c\times\{Q\in\partial D:|Q|>T\}$, where $0<\varepsilon<1$, T is a sufficiently large

Real number and M is a constant depending only on ε, D_c and T. The proof is complete.

Theorem **3.5.** Let $\{\mathcal{K}_m\}_{m=1}^{\infty}$ be the sequence of the polyharmonic fundamental solutions, and D be a Lipschitz graph domain in $\mathbb{R}^2$ with Lipschitz graph boundary ∂D. Then any $X=(x_1,x_2)$, $V=(v_1,v_2)\in\bar{D}$ and $X\neq V$,

$$\Delta_X\mathcal{K}_1(X,Y)=\Delta_Y\mathcal{K}_1(X,Y)=0 \qquad (3.9)$$

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And

$$\Delta_X \mathcal{K}_m\left(X,Y\right)=\Delta_Y \mathcal{K}_m(X,Y)=\mathcal{K}_{m-1}\left(X,Y\right) \tag{3.10}$$

Where $m \geq 2$, $\Delta_X = \sum_{j=1}^{2}\frac{\partial^2}{\partial x_j^2}$ and $\Delta_Y = \sum_{j=1}^{2}\frac{\partial^2}{\partial y_j^2}$.

Proof: By straightforward evaluation, we have $\Delta_X \mathcal{K}_1\left(X,Y\right)=\Delta_Y \mathcal{K}_1(X,Y)=0$. By The definition of $\mathcal{K}_m$, we have

$$S.P.\left[\mathcal{D}_m\right]\left(X,V\right)=C_m\sum_{l=0}^{2m-1}C_{m,l}(X,V)\left|V\right|^{2m-2-l} \tag{3.11}$$

Where C_m is a constant depending only on m, and the coefficient function $C_{m,l}$ Can be explicitly expressed by $P_l^{(1-m)}\left(X_{S^1}\cdot V_{S^1}\right)$, $Q_l^{(1-m)}\left(X_{S^1}\cdot V_{S^1}\right)$, $\left|X\right|^l$ and $\log\left|V\right|$. Then,

$$\Delta S.P.\left[\mathcal{D}_m\right]\left(X,V\right)=C_m\sum_{l=0}^{2m-1}\Delta C_{m,l}(X,V)\left|V\right|^{2m-2-l} \tag{3.12}$$

By Lemma 3.1, we have

$$\Delta\mathcal{K}_m-\mathcal{K}_{m-1}=S.P.\left[\mathcal{D}_{m-1}\right]-S.P.\left[\mathcal{D}_m\right] \tag{3.13}$$

For any $m \geq 2$. By Theorem 3.4. and (3.3), we have

$$\Delta\mathcal{K}_m=\mathcal{K}_{m-1} \text{ and } \Delta S.P.\left[\mathcal{D}_m\right]=S.P.\left[\mathcal{D}_{m-1}\right] \tag{3.14}$$

For sufficiently large $\left|V\right|$, and $m \geq 2$. By the symmetric of $\mathcal{K}_m$, we finish the proof.

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