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Anticipated Reflected Backward Doubly SDEs with Generators in Stochastic Lipschitz Condition

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ABSTRACT. *In this paper, we prove the well-posedness of the solutions to anticipated reflected backward doubly SDEs (ARBDSDEs for short) with stochastic Lipschitz coefficients.*

KEYWORDS: *backward douly stochastic differential equations, stochastic Lipschitz condition, reflected solution*

1. Introduction

In 1994, Pardoux and Peng [1] proposed a new type of stochastic differential equation called Backward Douly Stochastic Differential Equations (BDSDEs) and discovered the relation between BDSDEs and a type of quasilinear SPDEs. The BDSDEs are in the following general form:

$$Y_t = \xi_T + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B_s} - \int_t^T Z_s dW_s, t \in [0,T]. \tag{1}$$

Since then there has been massive applications of BDSDEs. Bahlali et al. [2] extended this type of equation to the reflected case (RBDSDEs) in 2009, which means the solution stays above a given obstacle. Luo et al. [3] extend the reflected doubly case to the anticipated condition (ARBDSDEs), where the coefficients depend on the future values of the solution. However, the above articles all consider equations in a uniform Lipshitz situation, which is hard to satisfy in realistic practice. Thus we consider the more relaxed stochasitc Lipschitz condition which has been discussed for BSDEs ([4]), BDSDEs ([5]) but has never been considered for ARBDSDEs. Based our hypothesis on coefficients, we derive the existence and uniqueness of the solution to the ARBDSDE, which is the following equation:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)})ds + K_T - K_t \\ \qquad + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)})d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \quad t \in [0,T]; \\ Y_t = \xi_t, \quad Z_t = \eta_t, \qquad t \in [T, T+K], \\ \int_0^T (Y_s - S_s)dK_s = 0,\ P-a.s., \text{ and } Y_s \geq S_s, s \in [0,T], \\ \text{where S is continuous,} \quad E[\sup_{0\leq t\leq T} e^{2\beta A(t)}(S_t^+)^2] < +\infty. \end{cases} \tag{2}$$

The rest of the paper is as follows: Notations, assumptions and definitions are stated in Section 2. Section 3 deals with the existence and uniqueness.

2. Notations, assumptions and definitions

2.1 Notations

In this section we introduce some basic notations. Let ($\Omega, \mathcal{F}, \mathbb{P}$) be a probability space and { $W_t, 0 \leq t \leq T$ } and { $B_t, 0 \leq t \leq T$ } be two mutually independent standard Brownian motions with values in $\mathbb{R}^d$ and $\mathbb{R}^l$ respectively. For each $t \in [0, T+K]$, we define $\mathcal{G}_t = \mathcal{G}_{0,t}^W \vee \mathcal{G}_{t,T+K}^B$, where for any process η, $\mathcal{G}_{s,t}^\eta = \sigma$ { $\eta_r - \eta_s, s \leq r \leq t$ } $\vee\ \mathcal{N}$ and $\mathcal{N}$ is the class of $\mathbb{P}$-null sets of $\mathcal{G}$. Note that the collection { $\mathcal{G}_t, 0 \leq t \leq T$ } is neither increasing nor decreasing, so it does not constitute a filtration. Similarly we define $\mathcal{F}_t = \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,T}^B$.

For any positive stochastic process $(a(t))_{t\geq 0}$, such that $a(t)$ is $\mathcal{G}_t$ adapted for $t \geq 0$, we define an increasing process $(A(t))_{t\geq 0}$ by setting $A(t) = \int_0^t a^2(s)ds$.

For every $\beta > 0$, let $L^2(\beta, a, T, \mathbb{R}^k)$ denote the set of k-dimensional $\mathcal{G}_T$-measurable random variables ξ such that $\|\xi\|_\beta^2 = \mathbb{E}(e^{\beta A(T)}|\xi|^2) < +\infty$.

Similarly, we denote by $L^2(\beta, a, [0,T], \mathbb{R}^k)$ and $L^{2,a}(\beta, a, [0,T], \mathbb{R}^k)$ the sets of k-dimensional jointly measurable processes $\{Y_t; t \in [0,T]\}$, such that Y_t is $\mathcal{G}_t$ adapted and which satisfy respectively

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$$\| Y\|_\beta^2 = \mathbb{E}\Big(\int_0^T e^{\beta A(s)} |Y_s|^2 \, ds\Big) < +\infty,$$
$$\| aY\|_\beta^2 = \mathbb{E}\Big(\int_0^T e^{\beta A(s)} |a(s)Y_s|^2 \, ds\Big) < +\infty.$$

Also, let $L_c^2(\beta, a, [0,T], \mathbb{R}^k)$ denote the sets of k-dimensional continuous processes $\{Y; t \in [0,T]\}$, such that Y_t is $\mathcal{G}_t$ adapted and satisfying $\| Y\|_{\beta,c}^2 = \mathbb{E}(\sup_{0\le t\le T} e^{\beta A(t)} |Y_t|^2) < +\infty$.

Since our equation is a reflected case, Y and K of the solution is 1-dimensional.

Note that the space $L^2(\beta, a, [0,T], \mathbb{R})$ with the norm $\|\cdot\|_\beta$ is a Banach space. So is the space

$M(\beta, a, T) = L^{2,a}(\beta, a, [0,T], \mathbb{R}) \times L^2(\beta, a, [0,T], \mathbb{R}^d)$, with the norm $\|(Y,Z)\|_\beta^2 = \| aY\|_\beta^2 + \| Z\|_\beta^2$. Also is the space

$M^c(\beta, a, [0,T]) = (L^{2,a}(\beta, a, [0,T], \mathbb{R}) \cap L_c^2(\beta, a, [0,T], \mathbb{R})) \times L^2(\beta, a, [0,T], \mathbb{R}^d)$, with the norm $\|(Y,Z)\|_{\beta,c}^2 = \| Y\|_{\beta,c}^2 + \| aY\|_\beta^2 + \| Z\|_\beta^2$.

Notice that if $b(t)$ and $a(t)$ are two jointly measurable, $\mathcal{G}_t$ adapted positive processes with $b > a$, then $L^2(\beta, b, [0,T], \mathbb{R}) \subset L^2(\beta, a, [0,T], \mathbb{R})$. Therefore $M^2(\beta, b, T) \subset M^2(\beta, a, T) \subset M^2(\beta, 0, T).$

2.2 Assumptions and definitions

In this paper, we will treat the case when ξ is $\mathcal{G}_t$-measurable.

For each $t \in [0,T]$, let

$f(t,\cdot,\cdot,\cdot,\cdot): \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times L^{2,a}(\beta, a, [t, T+K]; \mathbb{R}) \times L^2(\beta, a, [t, T+K]; \mathbb{R}^d) \to \mathbb{R},$

$g(t,\cdot,\cdot,\cdot,\cdot): \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times L^{2,a}(\beta, a, [t, T+K]; \mathbb{R}) \times L^2(\beta, a, [t, T+K]; \mathbb{R}^d) \to \mathbb{R}^l.$

We make the following hypotheses:

(H1) $f(t,\cdot,\cdot,\cdot,\cdot)$ and $g(t,\cdot,\cdot,\cdot,\cdot)$ are jointly measurable and $\mathcal{G}_t$-adapted, and there exist three nonnegative $\mathcal{F}_t$-adapted processes $\mu(t), \nu(t), u(t), t \in [0, T+K]$,

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and two constants $0<\alpha_1<1, 0\le\alpha_2<\dfrac{1}{M}$, *satisfying* $0<\alpha_1+\alpha_2 M<1$, such that for any $r,\bar{r}\in[T,T+K]$,

$$(t,y,z,\theta,\phi),(t,y',z',\theta',\phi')$$
$$\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times L^{2,a}(\beta,a,[t,T+K];\mathbb{R})\times L^2(\beta,a,[t,T+K];\mathbb{R}^d),$$

we have

$$|f(t,y,z,\theta_r,\phi_{\bar{r}}-f(t,y',z',\theta'_r,\phi'_{\bar{r}})|^2$$
$$\le\mu(t)(|y-y'|^2+E^{\mathcal{F}_t}|\theta_r-\theta'_r|^2)+\nu(t)(|z-z'|^2+E^{\mathcal{F}_t}|\phi_{\bar{r}}-\phi'_{\bar{r}}|^2),$$

$$|g(t,y,z,\theta_r,\phi_{\bar{r}}-g(t,y',z',\theta'_r,\phi'_{\bar{r}})|^2$$
$$\le u(t)(|y-y'|^2+E^{\mathcal{F}_t}|\theta_r-\theta'_r|^2)+\alpha_1|z-z'|^2+\alpha_2 E^{\mathcal{F}_t}|\phi_{\bar{r}}-\phi'_{\bar{r}}|^2.$$

(**H2**) For $t\in[0,T+K]$,

$$a^2(t)=\mu(t)+\nu(t)+u(t)\ge 1.$$

(**H3**)

$$E\int_0^T e^{\beta A(s)}\left|\frac{f(s,0,0,0,0)}{a(s)}\right|^2 ds<\infty,\ E\int_0^T e^{\beta A(s)}|g(s,0,0,0,0)|^2 ds<\infty.$$

Now we introduce two nonnegative continuous functions $\delta(\cdot)$ and $\zeta(\cdot)$ on [0,T] such that

(**a**1) There exists a constant $K\ge 0$ such that for all $s\in[0,T]$, $s+\delta(s)\le T+K;\qquad s+\zeta(s)\le T+K.$

(**a2**) For any process defined above, there exists a constant $M(a(.))\ge 0$ such that for any $\beta>0$, any $Y\in L^{2,a}(\beta,a,[0,T+K])$, $Z\in L^2(\beta,a,[0,T+K])$, $t\in[0,T]$,

we have

$$E\int_t^T e^{\beta A(s)}a^2(s)|Y_{s+\delta(s)}|^2 ds\le ME\int_t^{T+K}e^{\beta A(s)}a^2(s)|Y_s|^2 ds,$$
$$E\int_t^T e^{\beta A(s)}|Z_{s+\zeta(s)}|^2 ds\le ME\int_t^{T+K}e^{\beta A(s)}|Z_s|^2 ds.$$

ISSN 2616-5667 Vol.3, Issue 3: 68-73, DOI: 10.25236/AJETS.2020.030308

Definition2.1. (Y, Z, K) is a solution of ARBDSDE if $(Y,Z)\in M^c(\beta,a,T+K)$, $K_t(t\in[0,T])\in L_c^2([0,T],\mathbb{R})$ is a non-decreasing process starting at $K_0=0$ and satisfies equation (2).

3. Existence and uniqueness theorem

Now we give our main results.

Theorem 3.1. Assume that (a1)-(a2) and (H1)-(H3) hold. Given $(\xi,\eta)\in M^c(\beta,a,[T,T+K])$, and a continuous process S_s , where $E[\sup_{0\le t\le T} e^{2\beta A(t)}(S_t^+)^2]<+\infty$, and $S_T\le\xi_T$, then the Equation (2) has a unique solution (Y, Z, K).

We firstly deal with a special situation where the coefficients f and g do not contain the $Y_{t+\delta(t)}, Z_{t+\zeta(t)}$ part. Based on this result and our hypothesis, Theorem 3.1 obviously holds by the standard fixed point theorem, thus the most important step is the following proposition :

Proposition 3.1. Assume $E\int_0^T e^{\beta A(s)}[|\frac{f(s)}{a(s)}|^2+|g(s)|^2]ds<\infty.$ For β sufficiently large, given $\xi_T\in L^2(\beta,a,T;\mathbb{R})$, a continuous process S_s , where $E[\sup_{0\le t\le T} e^{2\beta A(t)}(S_t^+)^2]<+\infty$, and $S_T\le\xi_T$, then there exists a unique solution (Y,Z,K) of Eq.(2).

Proof: Since

$$E[|\xi_T|^2]\le E[e^{\beta A(T)}|\xi_T|^2]<+\infty,$$
$$E[\int_0^T|g(s)|^2ds]\le E[\int_0^T e^{\beta A(s)}|g(s)|^2]<+\infty,$$
$$E[|\int_0^T|f(s)|ds|^2]\le E[\Big(\int_0^T|\frac{f(s)}{a(s)}|^2 e^{\beta A(s)}ds\Big)\Big(\int_0^T a^2(s)e^{-\beta A(s)}ds\Big)] \quad \text{(Cauchy-Schwartz Inequality)}$$
$$\le\frac{1}{\beta}E[\int_0^T|\frac{f(s)}{a(s)}|^2 e^{\beta A(s)}ds]<+\infty,$$
$$E[\sup_{0\le t\le T}(S_t^+)^2]\le E[\sup_{0\le t\le T}e^{2\beta A(s)}(S_t^+)^2]<+\infty.$$

by Proposition C.1. in [6], which is the classical Snell envelope method, there is a unique solution (Y,Z,K) of (2) such that $(Y,Z)\in M^c(0,1,T)$.

ISSN 2616-5667 Vol.3, Issue 3: 68-73, DOI: 10.25236/AJETS.2020.030308

Now we prove that $(Y,Z)\in M^{c}(\beta,a,T)$. Indeed, using Ito's formula, B-D-G inequality and Cauchy-Schwartz inequality, we can get the following result, which is a slight change of Theorem 1 in [7]:

$$
\begin{aligned}
&E[\sup_{0\le t\le T} e^{\beta A(t)}|Y_t|^2+\int_0^T e^{\beta A(s)}(a^2(s)|Y_s|^2+|Z_s|^2)ds+|K_T|^2]\\
&\le CE[e^{\beta A(T)}|\xi_T|^2+\int_0^T e^{\beta A(s)}(|\frac{f(s)}{a(s)}|^2+|g(s)|^2)ds+\sup_{0\le t\le T} e^{2\beta A(t)}(S_t^{+})^2],
\end{aligned}
$$

where C is a positive constant depending on $\beta,\alpha_1,\alpha_2,M$.

Thus $(Y,Z)\in M^{c}(\beta,a,T)$. □

Using fixed point theorem, and our hypothesis (for instance see [3]) we can derive that Theorem 3.1 holds.

4. Conclusion

We proved the existence and unique result of the solution to ARBDSDE under stochastic Lipschitz situation. In the future we will explore the moment estimates of solutions of equations in this type.

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