ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

# Complete Moment Convergence for Weighted Sums of Extended Negatively Dependent Random Variables under Sublinear Expectation

**Yanchun Yi[a,*], Shuo Li[b]**

*College of Mathematics and Statistics, Hengyang Normal University, Hengyang, 421008, China*
*[a]laoyayyc@hotmail.com, [b]vincere0526@outlook.com*
*[*]Corresponding author*

***Abstract:*** *In probability theory, the convergence of dependent random variables is an extremely crucial concept, Complete moment convergence plays an important role in complete convergence, as it provides a more precise characterization of the convergence rate by incorporating moment conditions. Many scholars have conducted research on classical probability limit theory and achieved significant results. Due to the uncertainty in actual events, limit theory in classical probability spaces is no longer applicable. Therefore, in recent years, the limit theory on nonadditive probabilities or nonlinear expectations is a challenging problem that has attracted many researchers, many scholars have extended their research to sublinear expectation space. In this paper, the weighted sums of Extended Negatively Dependent random variables in the sublinear expectation space are studied. Through making full use of the characteristics of sublinear expectation, some inequalities and local Lipschitz functions, the sufficient conditions for their complete moment convergence are explored. Some conclusions in reference twelve are extended by it from the classical probability spaces to the sublinear expectations space, so some conclusions in the literature are improved.*

***Keywords:*** *Extended Negatively Dependence; Complete Moment Convergence; Weighted Sum; Sublinear Expectation*

## 1. Introduction

Limit theory in probability theory is a research focus. It can only study deterministic models in classical probability space, but in real life, many things are uncertain such as risk measurement, finance, and insurance. Since Peng Shige proposed the concept of sublinear expectation, this uncertainty problem has been solved [1]. It has effectively addressed uncertainty in probability. Since then, many scholars have begun to study it in sublinear expectation space and have achieved many results, for example, Li-Xin Zhang has obtained the law of iterated logarithm and the strong law of large numbers [2]. Shu-ting Pan studied several types of convergence [3]. Min-Zhou Xu focused negatively dependent random variables and obtained the conditions for their complete moment convergence [4 - 5]. Complete convergence and complete moment convergence are crucial areas on probability limit theory [6-9]. The concept of complete convergence was first proposed by MSU et al [10], and CHOW introduced the concept of complete moment convergence on the basis of complete convergence [11]. Subsequently, a great number of scholars have studied it in classical probability spaces. For example, Yan-Chun Yi have obtained the necessary and sufficient conditions of complete moment convergence by taking the weighted sums of Extended Negatively Dependent (END) random variable sequences as the research object [12]. In this paper, the complete moment convergence for weighted sums of END random variables under sublinear expectation is examined, aiming to establish the sufficient conditions for this convergence. Some of the conclusions in reference [12] are improved in the sublinear expectation space.

## 2. Preliminary Knowledge

### *2.1 The Relevant Definitions and Properties of Sublinear Expectations*

In this article, we utilize the theoretical knowledge proposed by Professor Peng Shige[1]. Suppose

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

that $H$ is a linear space which is consist of all real functions defined on a measurable space $(\Omega, F)$.It makes $\varphi(\xi_1,\cdots,\xi_n)\in H$ for any $\xi_1,\cdots,\xi_n \in H$, $\varphi$ belong to the linear space consisting of local Lipschitz functions denoted as $C_{l,lip}(R^n)$, and it satisfies :

$$\left|\varphi(s)-\varphi(t)\right| \le C\left(1+|s|^m+|t|^m\right)\left(|s-t|\right),\ \forall s,t\in R^n$$

For some $C>0, m\in \mathrm{N}$, relying on $\varphi$.

Definition 2.1[1] $E: H \to R=[-\infty,+\infty]$ is defined as a sublinear expectation, if for $\forall \xi,\eta\in H$, the following conditions are satisfied:

(1) Monotonicity: when $\xi\le\eta$, we have $E[\xi]\le E[\eta]$;

(2) Constant invariance: $E[C]=C$, for all $C\in R$;

(3) Secondary additivity: $E[\xi+\eta]\le E[\xi]+E[\eta]$;

(4) Homogeneity: $E[\lambda\xi]=\lambda E[\xi], \lambda\ge 0$.

The $(\Omega,H,E)$ is called sublinear expectation space. For a given sublinear expectation $E$, We define the conjugate expectation of $E$ as $\varepsilon(\xi)=-E(-\xi), \forall\xi\in H$.

Definition 2.2[2] Assume $\Gamma\subset F$, the function $V:\Gamma\to[0,1]$ is said to capacity, if the following conditions are met:

(1) $V(\phi)=0, V(\Omega)=1$;

(2) When $\xi_n\uparrow\xi$, we have $V(\xi_n)\uparrow V(\xi)$;when $\xi_n\downarrow\xi$, we have $V(\xi_n)\downarrow V(\xi)$, where $\xi,\xi_n\in\Gamma$;

(3) When $\xi\subset\eta$, we have $V(\xi)\le V(\eta), \forall\xi,\eta\in\Gamma$.

If for $A_i\in\Gamma, V\left(\bigcup_{i=1}^{\infty}A_i\right)\le\sum_{i=1}^{\infty}V(A_i)$, then $V$ is said to have additivity several times.

Definition 2.3 [2] Choquet Integral $C_V(\xi)=\int_0^{+\infty}V(\xi\ge s)ds+\int_{-\infty}^{0}\left(V(\xi\ge s)-1\right)ds$.

Definition 2.4[2] In sublinear expected space $(\Omega,H,E)$, if there exists a constant $h\ge 1$, such that $E\left[\prod_{i=1}^{n}\psi_i(\xi_i)\right]\le h\prod_{i=1}^{n}E\left[\left(\psi_i(\xi_i)\right)\right], n\ge 1$, where $\psi_i\in C_{l,lip}(R)$ is a non-negative function and neither decreasing nor increasing, then we call $\{\xi_i, i\ge 1\}$ a sequence of Extended Negatively Dependent (Abbreviated as END) random variables.

### *2.2 Lemmas*

When proving the main conclusion, the following lemmas are needed.

Lemma2.1 [3] (1) $C_r$ inequality: Given a column of random variables $\eta_1,\eta_2,\cdots,\eta_n$ in $(\Omega,H,E)$, the following inequality holds: for every positive number $r$,

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

$$E\left[\left|\eta_1+\eta_2+\cdots+\eta_n\right|^r\right]\le C_r\left\{E\left[\left|\eta_1\right|^r\right]+E\left[\left|\eta_2\right|^r\right]+\cdots+E\left[\left|\eta_n\right|^r\right]\right\},$$

Where $C_r=\begin{cases}1 & 0<r\le 1\\ n^{r-1} & r>1\end{cases}$.

(2) Markov-inequality: For $\forall\xi\in H, V\left(\left|\xi\right|\ge s\right)\le\frac{E\left(\left|\xi\right|^q\right)}{s^q},\forall s>0,q>0$.

(3) Jesen-inequality: If $f\left(x\right)$ is a convex function which is defined on $R$, suppose $E\left(\xi\right)$, $E\left[f\left(\xi\right)\right]$ are finite, then $E\left[f\left(\xi\right)\right]\ge f\left(E\left(\xi\right)\right)$.

Lemma2.2 Suppose there is a triangular array of random variables $\left\{\xi_{ij},1\le i,1\le j\le i\right\}$,

Which is defined on sublinear expected space $\left(\Omega,H,E\right)$. Let $\tau>0$, $\left\{a_i,i\ge 1\right\}$, $\left\{b_i,i\ge 1\right\}$ be a sequence of positive numbers. If for every $\varepsilon>0$,

$$\sum_{i=1}^{\infty}b_i\int_1^{\infty}V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|>\varepsilon a_i x^{1/\tau}\right)dx<\infty .$$

Then for every $\varepsilon>0$,

$$\sum_{i=1}^{\infty}b_i\,a_i^{-\tau}E\left\{\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|-\varepsilon a_i\right\}_+^{\tau}<\infty .$$

Proof: Let $\varepsilon>0$ be given, note that

$$\infty>\sum_{i=1}^{\infty}b_i\int_1^{\infty}V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|>2^{-1}\varepsilon a_i x^{1/\tau}\right)dx$$

$$\ge\sum_{i=1}^{\infty}b_i\int_1^{2^\tau}V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|>2^{-1}\varepsilon a_i x^{1/\tau}\right)dx$$

$$\ge\sum_{i=1}^{\infty}b_i\int_1^{2^\tau}V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|>\varepsilon a_i\right)dx$$

$$=\left(2^\tau-1\right)\sum_{i=1}^{\infty}b_i\,V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|>\varepsilon a_i\right)$$

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

Thus, we have

$$\sum_{i=1}^{\infty} b_i a_i^{-\tau} E\left\{\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|-\varepsilon a_i\right\}_+^{\tau}$$

$$=\sum_{i=1}^{\infty} b_i a_i^{-\tau}\int_0^{\infty} V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|-\varepsilon a_i > x^{1/\tau}\right)dx$$

$$=\sum_{i=1}^{\infty} b_i a_i^{-\tau}\int_0^{(\varepsilon a_i)^{\tau}} V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|-\varepsilon a_i > x^{1/\tau}\right)dx$$

$$+\sum_{i=1}^{\infty} b_i a_i^{-\tau}\int_{(\varepsilon a_i)^{\tau}}^{\infty} V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right|-\varepsilon a_i > x^{1/\tau}\right)dx$$

$$\le \varepsilon^{\tau}\sum_{i=1}^{\infty} b_i V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right| > \varepsilon a_i\right)+\sum_{i=1}^{\infty} b_i a_i^{-\tau}\int_{(\varepsilon a_i)^{\tau}}^{\infty} V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right| > x^{1/\tau}\right)dx$$

$$=\varepsilon^{\tau}\sum_{i=1}^{\infty} b_i V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right| > \varepsilon a_i\right)+\varepsilon^{\tau}\sum_{i=1}^{\infty} b_i \int_1^{\infty} V\left(\max_{1\le s\le i}\left|\sum_{j=1}^{s}\xi_{ij}\right| > \varepsilon a_i x^{1/\tau}\right)dx < \infty$$

## 3. Main Results

Theorem Let $\{\xi, \xi_j, j\ge 1\}$ be a sequence of END random variables in $(\Omega, H, E)$ satisfying $E(\xi_j)=0$ and $V(|\xi_j|>s)\le MV(|\xi|>s), \forall s\ge 0, j\ge 1$, where $M>0$ is a constant. We not only assume that

$$\alpha>1, \theta>0, p>\max\{0,(1-\theta)/(\alpha-1)\}, \beta>\max(-1,-1/\theta)$$

But also assume that

$p<2(1+\theta\beta)/(1+2\beta)$ and $\alpha>\max\{1,(3-\theta+2\beta)/[2(1+\theta\beta)]\}$, Where $\beta>-0.5$.

We denote $p(\alpha-1)+\theta$ as $\rho$. Suppose that there is a triangular array of constant numbers $\left\{A_{mj}=C_{mj} j^{\beta} m^{-(1+\theta\beta)/p}, 1\le m, 1\le j\le m\right\}$ that satisfies $|C_{mj}|\le C$ for any $1\le m, 1\le j\le m$, here $C>0$.

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

If $0<\tau<\begin{cases}(\rho-\theta)/(1+\theta\beta) & for \max\{-1,-1/\theta\}<\beta<-1/\rho \\ \rho & for \beta=-1/\rho \\ (\rho+p-\theta)/(1+\theta\beta-p\beta) & for \beta>-1/\rho\end{cases}$ (1)

And $\begin{cases}C_V|\xi|^{(\rho-\theta)/(1+\theta\beta)}<\infty & for \max\{-1,-1/\theta\}<\beta<-1/\rho \\ C_V|\xi|^{\rho}\log(1+|\xi|)<\infty & for \beta=-1/\rho \\ C_V|\xi|^{(\rho+p-\theta)/(1+\theta\beta-p\beta)} & for \beta>-1/\rho\end{cases}$ (2)

Then for every $\varepsilon>0$, $\sum_{m=1}^{\infty}m^{\alpha-2}E\left\{\max_{1\le h\le m}\left|\sum_{j=1}^{h}A_{mj}\xi_j\right|-\varepsilon\right\}_+^{\tau}<\infty$

Proof By the $C_r$ inequation and known conditions(1)(2), we have for every $m\ge1$, every $\varepsilon>0$,

$$E\left\{\max_{1\le h\le m}\left|\sum_{j=1}^{h}A_{mj}\xi_j\right|-\varepsilon\right\}_+^{\tau}<\infty.$$

Therefore, to prove the theorem holds, it only needs to be proven for sufficiently large integers $\hat{m}_0$,

$$\sum_{m=\hat{m}_0}^{\infty}m^{\alpha-2}E\left\{\max_{1\le h\le m}\left|\sum_{j=1}^{h}A_{mj}\xi_j\right|-\varepsilon\right\}_+^{\tau}<\infty \quad \text{for all } \varepsilon>0 \ .$$

Applying Lemma2.2 now, it is sufficient to prove

$$I=\sum_{m=\hat{m}_0}^{\infty}m^{\alpha-2}\int_1^{\infty}V\left(\max_{1\le h\le m}\left|\sum_{j=1}^{h}A_{mj}\xi_j\right|>\varepsilon t^{1/\tau}\right)dt<\infty \text{ for all } \varepsilon>0.$$

Since $A_{mj}=A_{mj}^{+}-A_{mj}^{-}$, thus, we may assume that $A_{mj}>0$.Given any $\varepsilon>0$, we choose some numbers as follow: $0<q<1$, some small $\sigma>0$, large integer $N\ge1$, and stipulate that, for all $1\le m, 1\le j\le m$, and all $t\ge1$,

$$\xi_{mj}^{(1,t)}=-m^{-\sigma}t^{q/\tau}I\left(A_{mj}\xi_j<-m^{-\sigma}t^{q/\tau}\right)+A_{mj}\xi_jI\left(\left|A_{mj}\xi_j\right|\le m^{-\sigma}t^{q/\tau}\right)+m^{-\sigma}t^{q/\tau}I\left(A_{mj}\xi_j>m^{-\sigma}t^{q/\tau}\right)$$

$$\xi_{mj}^{(2,t)}=\left(A_{mj}\xi_j-m^{-\sigma}t^{q/\tau}\right)I\left(m^{-\sigma}t^{q/\tau}\le A_{mj}\xi_j<\varepsilon t^{1/\tau}/(5N)\right),$$

$$\xi_{mj}^{(3,t)}=\left(A_{mj}\xi_j-m^{-\sigma}t^{q/\tau}\right)I\left(A_{mj}\xi_j>\varepsilon t^{1/\tau}/(5N)\right),$$

$$\xi_{mj}^{(4,t)}=\left(A_{mj}\xi_j-m^{-\sigma}t^{q/\tau}\right)I\left(\varepsilon t^{1/\tau}/(5N)\le A_{mj}\xi_j<-m^{-\sigma}t^{q/\tau}\right),$$

$$\xi_{mj}^{(5,t)}=\left(A_{mj}\xi_j+m^{-\sigma}t^{q/\tau}\right)I\left(A_{mj}\xi_j<-\varepsilon t^{1/\tau}/(5N)\right)$$

Then $I=\sum_{m=\hat{m}_0}^{\infty}m^{\alpha-2}\int_1^{\infty}V\left(\max_{1\le h\le m}\left|\sum_{j=1}^{h}A_{mj}\xi_j\right|>\varepsilon t^{1/\tau}\right)dt$

$$\le\sum_{l=1}^{5}\sum_{m=\hat{m}_0}^{\infty}m^{\alpha-2}\int_1^{\infty}V\left(\max_{1\le h\le m}\left|\sum_{j=1}^{h}\xi_{mj}^{(l,t)}\right|>\varepsilon t^{1/\tau}/5\right)dt:=I_1+I_2+I_3+I_4+I_5$$

Therefore, as long as we prove $I_l<\infty$ for $l=1,2,3,4,5$, then $I<\infty$ is proven.

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

For $I_1$, by $E\left(\xi_k\right)=0\ (k\geq 1)$ and known condition (2), we have

$$\sup\max_{1\leq h\leq m}\left|\sum_{j=1}^{h}E\xi_{mj}^{(1,t)}\right|$$

$$\leq\begin{cases}2\sum_{j=1}^{m}m^{\lambda\left(\frac{\rho-\theta}{1+\theta\beta}-1\right)}E\left|A_{mj}\xi_j\right|^{\frac{\rho-\theta}{1+\theta\beta}} & for\ \max\left\{-1,-\frac{1}{\theta}\right\}<\beta<-\frac{1}{\rho}\\ 2\sum_{j=1}^{m}m^{\lambda(\rho-1)}E\left|A_{mj}\xi_j\right|^{\rho} & for\ \beta=-\frac{1}{\rho}\\ 2\sum_{j=1}^{m}m^{\lambda\left(\frac{\rho-\theta+p}{1+\theta\beta-p\beta}-1\right)}E\left|A_{mj}\xi_j\right|^{\frac{\rho-\theta+p}{1+\theta\beta-p\beta}} & for\ \beta>-\frac{1}{\rho}\end{cases}$$

$$\leq\begin{cases}Cm^{-(\alpha-1)+\sigma\left(\frac{\rho-\theta}{1+\theta\beta}-1\right)} & for\ \max\left\{-1,-\frac{1}{\theta}\right\}<\beta<-\frac{1}{\rho}\\ Cm^{-(\alpha-1)+\sigma(\rho-1)}\log n & for\ \beta=-\frac{1}{\rho}\\ Cm^{-(\alpha-1)+\sigma\left(\frac{\rho+p-\theta}{1+\theta\beta-p\beta}-1\right)} & for\ \beta>-\frac{1}{\rho}\end{cases}$$

Choosing small $\sigma>0$ such that $\sigma\cdot\max\left\{\frac{\rho-\theta}{1+\theta\beta},\rho,\frac{\rho+p-\theta}{1+\theta\beta-p\beta}-1\right\}<\alpha-1$,

We get $\lim_{m\to\infty}\sup\max_{1\leq h\leq m}\left|\sum_{j=1}^{h}E\xi_{mj}^{(1,t)}\right|=0$ .

Therefore, if

$\sum_{m=\hat{m}_0}^{\infty}m^{\alpha-2}\int_1^{\infty}V\left(\max_{1\leq h\leq m}\left|\sum_{j=1}^{h}\left(\xi_{mj}^{(1,t)}-E\xi_{mj}^{(1,t)}\right)\right|>\varepsilon t^{1/\tau}/10\right)dt<\infty$ is proven,

Then $I_1<\infty$ is proven.

Choose $\gamma$ such that

$$\gamma>\max\left\{\begin{array}{l}2,\frac{\alpha}{\sigma},\frac{\tau}{1-q},\frac{2(\rho+p-\theta)}{1+\theta\beta},\frac{2(\rho+p-\theta)}{1+\theta\beta-p\beta},\frac{2(\rho+p-\theta)}{2-\theta}I(0<\theta<2),\\ \frac{2(\rho+p-\theta)}{2(1+\theta\beta)-p(1+2\beta)}I\left(\beta>-\frac{1}{2}\right)\end{array}\right\}$$

By using lemma 2.1and2.2, we can obtain

$$\sum_{m=\hat{m}_0}^{\infty}m^{\alpha-2}\int_1^{\infty}V\left(\max_{1\leq h\leq m}\left|\sum_{j=1}^{h}\left(\xi_{mj}^{(1,t)}-E\xi_{mj}^{(1,t)}\right)\right|>\varepsilon t^{1/\tau}/10\right)dt$$

$$\leq C\sum_{m=\hat{m}_0}^{\infty}m^{\alpha-2}\left(\log m\right)^{\gamma}\int_1^{\infty}t^{-\frac{\gamma}{\tau}}\left\{\sum_{j=1}^{m}E\left|\xi_{mj}^{(1,t)}-E\xi_{mj}^{(1,t)}\right|^{\gamma}+\left[\sum_{j=1}^{m}E\left(\xi_{mj}^{(1,t)}-E\xi_{mj}^{(1,t)}\right)^2\right]^{\gamma/2}\right\}dt$$

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

$$\le C\sum_{m=1}^{\infty} m^{\alpha-2}\left(\log m\right)^{\gamma}\int_1^{\infty} t^{-\frac{\gamma}{\tau}}\sum_{j=1}^{m} E\left|\xi_{mj}^{(1,t)}\right|^{\gamma} dt + C\sum_{j=1}^{m} m^{\alpha-2}\left(\log m\right)^{\gamma}\int_1^{\infty} t^{-\frac{\gamma}{\tau}}\left[\sum_{i=1}^{n} E\left(\xi_{mj}^{(1,t)}\right)^2\right]^{\frac{\gamma}{2}} dt$$

According the definition of $\xi_{mj}^{(1,t)}$ and the choosing of $\gamma$, we can get

$$C\sum_{m=1}^{\infty} m^{\alpha-2}\left(\log m\right)^{\gamma}\int_1^{\infty} t^{-\frac{\gamma}{\tau}}\sum_{j=1}^{m} E\left|\xi_{mj}^{(1,t)}\right|^{\gamma} dt \le \sum_{m=1}^{\infty} m^{\alpha-1-\lambda\gamma}\left(\log m\right)^{\gamma}\int_1^{\infty} t^{-\frac{p(1-q)}{\tau}} dt < \infty$$

The following proves

$$C\sum_{j=1}^{m} m^{\alpha-2}\left(\log m\right)^{\gamma}\int_1^{\infty} t^{-\frac{\gamma}{\tau}}\left[\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2\right]^{\frac{\gamma}{2}} dt < \infty,$$

The following three cases are going to be discussed:

1) $\max\{-1,-1/\theta\} < \beta < -1/\rho$

(a) If $(\rho-\theta)/(1+\theta\beta) \ge 2$ and $0<\theta<2$, then $E|\xi|^2<\infty$, therefore,

$$\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2 \le E\left(A_{mj}\xi_j\right)^2 \le C\sum_{j=1}^{m}\left(A_{mj}\right)^2$$

$$\le \begin{cases} Cm^{-2(1+\theta\beta)/p} & \text{for } \max\{-1,-1/\theta\}<\beta<-1/2 \\ Cm^{-(2-\theta)/p}\log n & \text{for } \beta=-1/2 \\ Cm^{\left[-1-2\beta+2(1+\theta\beta)/p\right]} & \text{for } \beta>-1/2 \end{cases}$$

(b) If $(\rho-\theta)/(1+\theta\beta) \ge 2$ and $\theta \ge 2$, in which case, we have that $E|\xi|^2<\infty$ and $\beta > -1/2$, thus,

$$\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2 \le E\left(A_{mj}\xi_j\right)^2 \le C\sum_{j=1}^{m}\left(A_{mj}\right)^2 \le Cm^{-\left[-1-2\beta+2(1+\theta\beta)/p\right]}$$

(c) If $(\rho-\theta)/(1+\theta\beta)<2$, by $\beta<-1/\rho$, we have $p\beta(\alpha-1)/(1+\theta\beta)<-1$, therefore, we get

$$\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2 = \sum_{j=1}^{m} E\left(A_{mj}\xi_j\right)^2 I\left(\left|A_{mj}\xi_j\right| \le m^{-\sigma}t^{q/\tau}\right) + \left(n^{-\sigma}t^{q/\tau}\right)^2 I\left(\left|A_{mj}\xi_j\right| > m^{-\sigma}t^{q/\tau}\right)$$

$$\le \left(m^{-\sigma}t^{q/\tau}\right) \le \left(m^{-\sigma}t^{q/\tau}\right)^{2-p(\alpha-1)/(1+\theta\beta)}\sum_{j=1}^{m} E\left|A_{mj}\xi_j\right|^{p(\alpha-1)/(1+\theta\beta)}$$

$$\le Cm^{-(\alpha-1)-\sigma\left[2-p(\alpha-1)/(1+\theta\beta)\right]} t^{q\left[2-p(\alpha-1)/(1+\theta\beta)\right]/\tau}$$

2) $\beta=-1/\rho$

(a) If $\rho \ge 2$ and $0<\theta<2$, then

$$\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2 \le E\left(A_{mj}\xi_j\right)^2 \le C\sum_{j=1}^{m}\left(A_{mj}\right)^2$$

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

$$\leq \begin{cases} Cm^{-2(1+\theta\beta)/p} & for \max\{-1,-1/\theta\}<\beta<-1/2 \\ Cm^{-(2-\theta)/p}\log n & for\ \beta=-1/2 \\ Cm^{[-1-2\beta+2(1+\theta\beta)/p]} & for\ \beta>-1/2 \end{cases}$$

(b) If $\rho \geq 2$ and $\theta \geq 2$, then

$$\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2 \leq E\left(A_{mj}\xi_j\right)^2 \leq C\sum_{j=1}^{m}\left(A_{mj}\right)^2 \leq Cm^{-[-1-2\beta+2(1+\theta\beta)/p]}$$

(c) If $\rho < 2$, since $\beta = -1/\rho$, by known condition (2), we have

$$\sum_{i=1}^{n} E\left(\xi_{ni}^{(1,t)}\right)^2 \leq \left(m^{-\sigma}t^{q/\tau}\right)^{2-(\rho-\theta)/(1+\theta\beta)} \sum_{j=1}^{m} E\left|A_{mj}\xi_j\right|^{\rho}$$

$$\leq Cm^{-(\alpha-1)-\sigma(2-\rho)}(\log m)t^{q(2-\rho)/\tau}.$$

3) $\beta > -1/\rho$

(a) If $(\rho+p-\theta)/(1+\theta\beta-p\beta) \geq 2$ and $0<\theta<2$, then

$$\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2 \leq E\left(A_{mj}\xi_j\right)^2 \leq C\sum_{j=1}^{m}\left(A_{mj}\right)^2$$

$$\leq \begin{cases} Cm^{-2(1+\theta\beta)/p} & for \max\{-1,-1/\theta\}<\beta<-1/2 \\ Cm^{-(2-\theta)/p}\log m & for\ \beta=-1/2 \\ Cm^{[-1-2\beta+2(1+\theta\beta)/p]} & for\ \beta>-1/2 \end{cases}$$

(b) If $(\rho+p-\theta)/(1+\theta\beta-p\beta) \geq 2$ and $\theta \geq 2$, then

$$\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2 \leq E\left(A_{mj}\xi_j\right)^2 \leq C\sum_{j=1}^{m}\left(A_{mj}\right)^2 \leq Cm^{-[-1-2\beta+2(1+\theta\beta)/p]}$$

(c) If $(\rho+p-\theta)/(1+\theta\beta-p\beta) < 2$, by $\beta > -1/\rho$, we have

$p\beta(\alpha-1)/(1+\theta\beta) > -1$, therefore, combing the known conditions, we get

$$\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2 \leq \left(m^{-\sigma}t^{q/\tau}\right)^{2-(\rho-\theta+p)/(1+\theta\beta-p\beta)} \sum_{j=1}^{m} E\left|A_{mj}\xi_j\right|^{(\rho-\theta+p)/(1+\theta\beta-p\beta)}$$

$$\leq Cm^{-(\alpha-1)-\sigma[2-(\rho-\theta+p)/(1+\theta\beta-p\beta)]}t^{q[2-(\rho-\theta+p)/(1+\theta\beta-p\beta)]/\tau}$$

Based on the above discussion and the selection of $\gamma$, .we have obtained

$$C\sum_{m=1}^{\infty} m^{\alpha-2}(\log m)^{\gamma}\int_1^{\infty} t^{-\frac{\gamma}{\tau}}\left[\sum_{j=1}^{m} E\left(\xi_{mj}^{(1,t)}\right)^2\right]^{\frac{\gamma}{2}} dt < \infty$$

In this way, $I_1 < \infty$ is proven.

For $I_2$, by the definition of $\xi_{mj}^{(2,t)}$ and Markov-inequality, we can obtain for $m \geq N$

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

$$V\left(\max_{1\le k\le m}\left|\sum_{j=1}^{k}\xi_{mj}^{(2,t)}\right|>\varepsilon t^{1/\tau}/5\right)=V\left(\sum_{j=1}^{k}\xi_{mj}^{(2,t)}>\varepsilon t^{1/\tau}/5\right)$$

$$\le V\left(\textit{there are at least } N \textit{ values of } j\in\{1,2,\cdots,m\}\textit{ such that } A_{mj}\xi_j>m^{-\sigma}t^{q/\tau}\right)$$

$$\le\sum_{1\le j_1\le\cdots\le j_N}V\left(A_{mj_1}\xi_{j_1}>m^{-\sigma}t^{q/\tau},\cdots,A_{mj_N}\xi_{j_N}>m^{-\sigma}t^{q/\tau}\right)$$

$$\le\sum_{1\le j_1\le\cdots\le j_N}MV\left(A_{mj_1}\xi_{j_1}>m^{-\sigma}t^{q/\tau}\right)\cdots V\left(A_{mj_N}\xi_{j_N}>m^{-\sigma}t^{q/\tau}\right)$$

$$\le M\left(\sum_{j=1}^{m}V\left(A_{mj}\xi_j>m^{-\sigma}t^{q/\tau}\right)\right)^N$$

$$\le\begin{cases}M\left(\sum_{j=1}^{m}\left(m^{\sigma}t^{-\frac{q}{\tau}}\right)^{\frac{\rho-\theta}{1+\theta\beta}}E\left|A_{mj}\xi_j\right|^{\frac{\rho-\theta}{1+\theta\beta}}\right)^N & \textit{for } \max\left\{-1,-\frac{1}{\theta}\right\}<\beta<-\frac{1}{\rho}\\ M\left(\sum_{j=1}^{m}\left(m^{\sigma}t^{-\frac{q}{\tau}}\right)^{\rho}E\left|A_{mj}\xi_j\right|^{\rho}\right)^N & \textit{for } \beta=-\frac{1}{\rho}\\ M\left(\sum_{j=1}^{m}\left(m^{\sigma}t^{-\frac{q}{\tau}}\right)^{\frac{\rho-\theta+p}{1+\theta\beta-p\beta}}E\left|A_{mj}\xi_j\right|^{\frac{\rho-\theta+p}{1+\theta\beta-p\beta}}\right)^N & \textit{for } \beta>-\frac{1}{\rho}\end{cases}$$

$$\le\begin{cases}Cm^{\left[-(\alpha-1)+\frac{\sigma(\rho-\theta)}{1+\theta\beta}\right]N}t^{\frac{-Nq(\rho-\theta)}{\tau(1+\theta\beta)}} & \textit{for } \max\left\{-1,-\frac{1}{\theta}\right\}<\beta<-\frac{1}{\rho}\\ Cm^{\{-(\alpha-1)+\sigma\rho\}N}t^{\frac{-Nq\rho}{\tau}}(\log m)^N & \textit{for } \beta=-\frac{1}{\rho}\\ Cm^{\left[-(\alpha-1)+\frac{\sigma(\rho-\theta+p)}{1+\theta\beta-p\beta}\right]N}t^{\frac{-Nq(\rho-\theta+p)}{\tau(1+\theta\beta-p\beta)}} & \textit{for } \beta>-\frac{1}{\rho}\end{cases}$$

Because of $\sigma\cdot\max\left\{\frac{\rho-\theta}{1+\theta\beta},\rho,\frac{\rho-\theta+p}{1+\theta\beta-p\beta}-1\right\}<\alpha-1$,

We can take a sufficiently large positive $N$ such that

$$N\left[-(\alpha-1)+\sigma\cdot\max\left\{\frac{\rho-\theta}{1+\theta\beta},\rho,\frac{\rho-\theta+p}{1+\theta\beta-p\beta}\right\}\right]<-\alpha \quad \text{and}$$

$$N\frac{q}{\tau}\cdot\min\left\{\frac{\rho-\theta}{1+\theta\beta},\rho,\frac{\rho-\theta+p}{1+\theta\beta-p\beta}\right\}>1$$

Thus, we get $I_2<\infty$

For $I_3$, since

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

$$I_3 \le \sum_{m=1}^{\infty} m^{\alpha-2} \int_1^{\infty} V\left(\bigcup_{j=1}^{\infty}\left(\left|A_{mj}\xi_j\right| > \frac{\varepsilon t^{1/\tau}}{5N}\right)\right)dt$$

$$\le \sum_{m=1}^{\infty} m^{\alpha-2} \int_1^{\infty} V\left(\bigcup_{j=1}^{\infty}\left(\left|\xi_j\right| > \frac{\varepsilon}{5ND} t^{1/\tau} j^{-\beta} m^{(1+\theta\beta)/p}\right)\right)dt$$

$$\le C\int_1^{\infty}\left\{\sum_{m=1}^{\infty} m^{\alpha-2} \sum_{j=1}^{m} V\left(\left|\xi\right| > \frac{\varepsilon}{5ND} t^{1/\tau} j^{-\beta} m^{(1+\theta\beta)/p}\right)\right\}dt \tag{3}$$

Let $\mu = y^{-\beta} x^{(1+\theta\beta)/p}, \upsilon = y,$ then $x = \mu^{p(1+\theta\beta)} \upsilon^{p\beta/(1+\theta\beta)}, y = \upsilon$.

Therefore, we have $\sum_{m=1}^{\infty} m^{\alpha-2} \sum_{j=1}^{m} V\left(\left|\xi\right| > \frac{\varepsilon}{5ND} t^{1/\tau} j^{-\beta} m^{(1+\theta\beta)/p}\right)$

$$\le C\int_1^{\infty}\int_1^{x} V\left(\left|\xi\right| > \frac{\varepsilon}{5ND} t^{1/\tau} y^{-\beta} x^{(1+\theta\beta)/p}\right)dxdy$$

$$\le \frac{pC}{1+\theta\beta}\int_1^{\infty} d\mu \int_1^{\mu^{p/(1+\theta\beta-p\beta)}} \mu^{(\rho-\theta)/(1+\theta\beta)-1} \upsilon^{\beta(\rho-\theta)/(1+\theta\beta)} V\left(\left|\xi\right| > \frac{1}{D} t^{1/\tau} \mu\right)d\upsilon$$

$$\le \begin{cases} C\int_1^{\infty} \mu^{\frac{\rho-\theta}{1+\theta\beta}} V\left(\left|\xi\right| > \frac{\varepsilon}{5ND} t^{1/\tau}\mu\right)d\mu & for\ \max\left\{-1, -\frac{1}{\theta}\right\} < \beta < -\frac{1}{\rho} \\ C\int_1^{\infty} \mu^{\rho-1}\left(\log\mu\right) V\left(\left|\xi\right| > \frac{\varepsilon}{5ND} t^{1/\tau}\mu\right)d\mu & for\ \beta = -\frac{1}{\rho} \\ C\int_1^{\infty} \mu^{\frac{\rho+p-\theta}{1+\theta\beta-p\beta}} V\left(\left|\xi\right| > \frac{\varepsilon}{5ND} t^{1/\tau}\mu\right)d\mu & for\ \beta > -\frac{1}{\rho} \end{cases}$$

$$\le \begin{cases} Ct^{-(\rho-\theta)/\tau(1+\theta\beta)} E\left|\xi\right|^{-(\rho-\theta)/\tau(1+\theta\beta)} & for\ \max\left\{-1, -1/\theta\right\} < \beta < -1/\rho \\ Ct^{-\rho/\tau} E\left|\xi\right|^{\rho} \log\left(1+\left|\xi\right|\right) & for\ \beta = -1/\rho \\ Ct^{-(\rho-\theta+p)/[\tau(1+\theta\beta-p\beta)]} E\left|\xi\right|^{(\rho-\theta+p)/(1+\theta\beta-p\beta)} & for\ \beta > -1/\rho \end{cases} \tag{4}$$

Therefore, by known conditions and (3)(4), $I_3 < \infty$ is proved. According to the proof method of $I_2 < \infty$, $I_3 < \infty$, we can also be proven that $I_4 < \infty$, $I_5 < \infty$ .Thus, the theorem is proved..

## 4. Conclusion

We have obtained the sufficient conditions for the convergence of weighted and complete moments of END random variables in subline expectation spaces, which extends the conclusions existing in reference [12].


## Acknowledgements

This research is supported by Hunan Provincial Natural Science Foundation of China (2023JJ50092).


## References


*[1] Peng Shige. G-expectation, G-Brownian motion and related stochastic calculus of Itos type[J].*

ISSN 2616-5805 Vol. 6, Issue 2: 102-112, DOI: 10.25236/AJMS.2025.060214

*Stochastic analysis and application, 2007, 2(4):541-567.*
*[2] Zhang L X. Strong limit theorems for extended independent random variables and extended negatively dependent random variables under sub-linear expectations[J]. Acta Mathematica Scientia, 2022, 42(2): 467-490.*
*[3] Pan Shuting. Convergence of several types of dependent sequences in sublinear expected spaces [D]. Anqing: Anqing Normal University, 2024.*
*[4] Mingzhou Xu and Zhenyu Xie, Complete Moment Convergence For Weighted Sums of Negatively Dependent Random Variables Under Sub–Linear Expectations [J]. Journal of mathematical inequalities, 2025, 19(1):211-221.*
*[5] Mingzhou Xu. On the complete moment convergence of moving average processes generated by negatively dependent random variables under sub-linear expectations [J].AIMS Mathematics, 2024, 9(2): 3369–3385.*
*[6] Sun Peiyu. Limit properties of several types of dependent sequences under sublinear expectation [D]. Changchun: Jilin University, 2023.*
*[7] Huang W H, Wu P Y. Strong law of large numbers under moment restrictions in sublinear expectation spaces[J]. Communications in Statistics-Theory and Methods, 2022, 51(24): 8671-8683.*
*[8] Wu Q Y, Lu J F. Another form of chover's law of the iterated logarithm under sub-linear expectations[J]. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 2020, 114(1): 1-22.*
*[9] Wu Q Y, Jiang Y Y. Strong law of large numbers and chover's law of the iterated logarithm under sublinear expectations[J]. Journal of Mathematical Analysis Applications, 2018, 460(1): 252-270.*
*[10] Hsu P L, Robbins H. Complete convergence and the law of large numbers[J].Proceedings of the National Academy of Sciences of the United States of America.1947, 33(2):25-31.*
*[11] Chow Y S. On the rate of moment convergence of sample sums and extremes[J].Bulletin of the Institute of Mathematics Academia Sinica, 1988, 16(3):177-201.*
*[12] Yi Y C, Qiu D H, Chen P Y. Complete moment convergence for weighted sums of extended negatively dependent random variables [J]. Communications in Statistics-Theory and Methods, 2017, 46(20): 10189-10202.*