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Dynamic Attention Allocation in Risky Asset Market

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Abstract: *This paper introduces a noisy rational expectation model that incorporates the dynamics between information acquisition and attention allocation. The model, differentiated from standard frameworks, allows investors to endogenously choose their attention allocation to multiple noisy signals, enabling an analysis of the dynamic interaction. A significant aspect of the model is the correlation between investors' attention allocation and the anticipated value of the terminal payoff under a risk-neutral measure. The model demonstrates that the marginal cost of attention increases with attention allocation, and this cost is proportional to the expected value of a certain variance under the risk-neutral measure.*

Keywords: *Endogenous Attention Allocation, Risk-Neutral Measure, Rational Expectation*

1. Introduction

The financial market is an essential pillar of modern economies, providing an efficient mechanism for allocating resources and facilitating capital investment. Asset pricing and market volatility are among the most critical phenomena shaping the financial market dynamics. Understanding the factors that affect asset pricing and market volatility is crucial for policymakers, investors, and researchers alike.

Information acquisition in financial markets has gained increasing attention. The interaction between information acquisition and market volatility is vital to understanding the behavior of asset prices and volatility. As investors continuously update their portfolio management strategies based on available information, asset pricing and corresponding volatility are inevitably impacted. Conversely, market volatility can influence investors' motivations to acquire information. The early-stage literature on information and asset pricing often considers a static decision about obtaining information[1-3]. However, some further studies filled a gap in the literature by examining the dynamic nature of information acquisition and its relationship with asset pricing and market volatility.

One of the key insights from the literature on rational expectations models is that the level of information available to investors and their ability to process this information are crucial determinants of asset prices[4-6]. In a noisy rational expectations model, investors receive noisy signals about the payoff of a risky asset and update their beliefs accordingly[7-9].

On the other hand, the literature on attention allocation in financial markets has shown that investors have limited attention and face trade-offs when deciding how to allocate their attention across different sources of information[10-11]. Our model departs from the standard rational expectation framework by allowing investors to choose the optimal attention allocated to multiple noisy signals endogenously. This approach will enable us to analyze the dynamic interaction between information acquisition, market volatility, and attention allocation.

Our study contributes a noisy rational expectation model incorporating endogenous dynamics between information acquisition and attention allocation.

2. Model Framework

In the present model, we consider a continuum of representative investors within the range $[0,1]$, standing for many identical investors. These investors engage in trading a risk-free asset and a risky one. To increase the exposition of the model, we start from a discrete-time setting with $t_i = (i-1)\Delta t$ and approach the continuous-time limit by letting $\Delta t \to 0$ or $I := T/\Delta t \to \infty$. The model's primary distinctions from a conventional rational expectation framework arise from two aspects. First,

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information acquisition is endogenous, and the risky asset's payoff may not conform to a standard normal distribution. Second, investors must determine the optimal attention allocated to signals in addition to their investment choices within the traditional asset pricing framework.

2.1 Financial Market

We denote the liquidation value of the risky asset as I when an investor ceases trading, which is assumed to be a constant for simplicity. Representative investors, identical in preferences, could trade risky and risk-free assets in the financial market within the time range $[0, T]$. The risky asset generates a final stochastic payoff x at time T with prior belief $F(x)$ while no cash flows are derived before the time $T - \Delta t$.

Although the final payoff is realized on time T, trading could happen among investors in any period t_i. Define p_i as the price of the risky asset in the time t_i and the return on the risky asset between two consecutive trading periods as $r_i = p_{i+1} - p_i$. The information is incomplete since investors cannot observe the final payoff x directly. Consequently, the investors trade with noise which makes the price not fully reflect the final payoff.

Then we assume the exogenous asset supply in time t_i is denoted by n_i. For the tractability of our model, the increment of n_i is set to follow a random walk:

$$n_{i+1} - n_i \sim \mathcal{N}(0, \sigma^2 \Delta t) \tag{1}$$

2.2 Information Structure

In Kyle's paper[5], investors initially possess the same prior belief $F(x)$ regarding the final payoff. However, unlike his approach, we now assume that investors receive multi-dimensional independent signals. The simplest case is that the investors will receive $J \in \mathcal{N}^+$ independent signals to update their beliefs, and their choice of corresponding attention determines the precision of these signals.

During $[t, t + \Delta t]$, an arbitrary investor $k \in \mathcal{N}$ could choose the attention $a_k^j(t)$ on the j-th signal about the final payoff x. We assume the signal denoted by $s_k^j(t)$ can be considered a perturbation of x with an error ϵ_t following a normal distribution with mean 0. The precision is controlled by the corresponding attention allocated[6], and signals among different investors are independent across all trading periods. In particular, the investor k receives the following signals in period $[t, t + \Delta t]$:

$$s_k^j(t + \Delta t) = x + \epsilon_k^j(t + \Delta t)\epsilon_k^j(t + \Delta t) \sim \mathcal{N}(0, e^{-\alpha_k a_k^j(t)} \Delta t) \tag{2}$$

Economic literature often sets an upper bound of the total volume of investors' attention as a capacity condition[8], which is sensible in a general macroeconomy setting but not in the financial market. Instead, we assume that the investors, whose attentions are so homogeneous that we only need to focus on the total amount, must pay a cost to obtain signals. The cost for the investor k during $[t, t + \Delta t]$ is $C(\sum_j a_k^j(t))\Delta t$ where the function $C(\cdot) \in \mathcal{C}^2$ is an increasing and convex function since the marginal cost of attention is increasing intuitively.

2.3 Learning and Equilibrium

A Bayesian procedure characterizes the learning process in this model, and the information received is divided into private and public. The series of these signals are regarded as private information, and the series of the risky asset price is public.

For the investork, the private information in period $[0, T]$ is denoted by $\{\vec{s}_k(t_1), \vec{s}_k(t_2), \ldots, \vec{s}_k(T)\}$ where $\vec{s}_k(t_i) = \left(s_k^1(t_i), s_k^2(t_i), \ldots, s_k^J(t_i)\right)$. At the same time, the public information is denoted as $\{p_0, p_1, \ldots, p_I\}$. Within the model, investors use both private and public information to update their beliefs of the final payoffx.

In addition, we need to consider market-clearing conditions characterizing the risky asset's price to achieve equilibrium. In other words, the aggregate trading demand of investors should be equal to the residual supply of the risky asset:

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$$\sum_k \theta_k(t) = n_t \tag{3}$$

where $\theta_k(t)$ represents the number of units of risky assets the investor k holds during $[t, t + \Delta t]$.

2.4 Investor Choice

For the investor k, the initial wealth is defined as $W_k(0)$, and the deterministic labor income in trading period $[t, t + \Delta t]$ is $w_k(t)$. Investors have to make choices about portfolio management and attention allocation. Then the investor k left $W_k(t) + w_k(t) - p_t\theta_k(t) - C(\sum_j a_k^j(t))\Delta t$ in the risk-free asset.

Considering the utility function, we assume all investors are independent and identical about the utility function $U(\cdot)$ of the final wealth in the time T. Thus, the investor k's expected utility is

$$E_k[U(W_k(T))|\mathcal{F}_t] \tag{4}$$

where $\mathcal{F}_t$ is the filtration up to the current time t.

Therefore, for investor k, the optimization problem in time t is:

$$\max_{\theta_k(t), a_k^1(t), a_k^2(t), \ldots, a_k^J(t)} E_k[U(W_k(T))|\mathcal{F}_t]$$
$$s.t.\, W_k(t + \Delta t) = W_k(t) + w_k(t) - p_t\theta_k(t) - C(\sum_{j=1}^{J} a_k^j(t))\Delta t \tag{5}$$

3. Theoretical Solutions

The series of residual supply $\{n_0, n_1, \ldots n_I\}$ and the true value of the final payoff x are exogenous. In contrast, the attention choice $\{a_k^1(t), a_k^2(t), \ldots, a_k^J(t)\}$, the risky asset price p_t, and the portfolio choice $\theta_k(t)$, are endogenous. We characterize the equilibrium of the financial market in a sequential procedure with three stages:

1) Investors make the attention and portfolio choices.

2) Given the investors' choices, the market-clearing condition is able to solve the asset demand as a function of the final payoff and the residual supply.

3) Use a dynamic allocation system to describe the movement of the risky asset's price and the volatility under the risk-neutral measure.

We assume the utility function is the constant absolute risk aversion (CARA) function of the final wealth. The risk aversion parameter is $\rho > 0$ and identical to all investors:

$$\forall k \in \mathcal{N}^+, U(W_k(T)) = -exp\{-\rho_k W_k(T)\} \tag{6}$$

3.1 Backward Induction

Considering the trading period is a finite interval $[0, T]$, backward induction is helpful to solve our model. It's straightforward that the investor should not allocate any attention during $[t_{I-1}, T]$ due to the information cost since the information realized after $t = t_{I-1}$ does not affect the final wealth because of the fact that there is no more decision to be made. Thus, the problem for the investor k is reduced to choose $\theta_k(t_{I-1})$ to maximize the expected utility:

$$\max_{\theta_k(t_{I-1})} \int_{(0,\infty)^I} -exp\{-\rho_k[\theta_k(t_{I-1})(x - p_{I-1}) + w_k(T)]\}\, dF_k(x|p_0, \ldots, p_{I-1}, s_k^1(t_{I-1}), \ldots, s_k^J(t_{I-1})) \tag{7}$$

where $F_k(x|\cdot)$ is the posterior belief about the payoff.

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We would assume the number of investors is large enough so that, at any time t, the risky asset price does not depend on any particular private signals. Mathematically, our assumption means the conditional distribution of x on p_i is independent from the series of signals. Also, since signals are normal random variables and independent with each other and reflecting the same fixed x at any specific moment t_i, we can use Bayes' law to combine them into an aggregate signal[12] $s_k(t_i)$, which is characterized by:

$$s_k(t_i) = x + \epsilon_k(t_i)\epsilon_k(t_i) \sim \mathcal{N}(0, \beta_k(t_i)\Delta t)\beta_k(t_i) = \frac{e^{-\alpha_k \sum_{j=1}^{J} a_k^j(t_i)}}{\sum_{j=1}^{J} e^{-\alpha_k a_k^j(t_i)}} \tag{8}$$

This method is also known as the weighted average method, the maximum likelihood estimate with scaling given the normal and independent noises in the signals, where the weights are inversely proportional to the variances of noises. The intuition behind this method is that signals with smaller variances are given more weight in the aggregate signal since they are more reliable measurements of the underlying x.

Considering that the aggregate private signal provides information independent of public information, we can rewrite Equation (7) as

$$\max_{\theta_k(t_{I-1})} \int_{(0,\infty)^I} -\exp\{-\rho_k[\theta_k(t_{I-1})(x - p_{I-1}) + w_k(T)]\}\, \phi\{s_k(t_{I-1}); x, \beta_k(t_{I-1})\Delta t\}\, dF_k(x|p_0, \dots, p_{I-1}) \tag{9}$$

where $\phi\{s_k(t_{I-1}); x, \beta_k(t_{I-1})\Delta t\}$ represents the probability density function of a normal variable with mean x and variance $\beta_k(t_{I-1})\Delta t$ at point $s_k(t_{I-1})$. Now assume the Equation (9) is denoted by $J(\theta_k(t_{I-1}))$ and compute the derivative of it:

$$\frac{dJ(\theta_k(t_{I-1}))}{d\theta_k(t_{I-1})} = -\int_{(0,\infty)^I} \rho_k(x - p_{I-1})(x - p_{I-1}) f(\theta_k(t_{I-1}))\phi\{s_k(t_{I-1}); x, \beta_k(t_{I-1})\Delta t\}\, dF_k(x|p_0, \dots, p_{I-1}) \tag{10}$$

where $f(\theta_k(t_{I-1})) = exp\{-\rho_k[\theta_k(t_{I-1})(x - p_{I-1}) + w_k(T)]\}$.

Upon substituting the formulation of the normal distribution, it emerges that the coefficient of x is tantamount to $-\rho_k + s_k(t_i)\sum_{j=1}^{J} a_k^j(t_i)$. The merit of this formulation lies in its embodiment as a linear amalgamation of the investor's portfolio selection and the cumulative private signal. Consequently, even though investors receive disparate signals contingent upon their individual attention choices, the optimization conundrums they confront adhere to a uniform paradigm.

Given the exponential nature of the component of x, the first-order condition with respect to $\theta_k(t_{I-1})$ is employed, leading to the discovery that the coefficient remains invariant across diverse investors. This outcome, which coheres with the aforementioned linear characteristic, intimates that the demand of investor i for the risky asset is separable in private signals, as postulated by the existing literature[13].

3.2 Portfolio Choice

For $i \in \{1,2,\dots,I-1\}$, the maximization involves attention allocation. With dynamic programming, we can rewrite the object as:

$$\begin{aligned} E_k[U(W_k(t_i))|\mathcal{F}_{t_{i-1}}] &= E_k[U(W_k(t_{i-1}) + w_k(t_{i-1}) - p_{t_{i-1}}\theta_k(t_{i-1}) - C[\sum_{j=1}^{J} a_k^j(t_i)]\Delta t)|\mathcal{F}_{t_{i-1}}] \\ &= -e^{\rho_k(W_k(t_{i-1}) + w_k(t_{i-1}))} E_k[e^{-\rho_k(p_{t_{i-1}}\theta_k(t_{i-1}) + C[\sum_{j=1}^{J} a_k^j(t_i)]\Delta t)}|\mathcal{F}_{t_{i-1}}] \end{aligned} \tag{11}$$

For an arbitraty investor i, the demand for the risky asset in period $t = \tau$ is articulated as the aggregate of a universally accessible public signal and the investor's own comprehensive private signal. This is mathematically represented as:

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$$\theta_{i,\tau} = \theta_i(\tau) + \frac{s_i(\tau)\sum_{j=1}^{J} a_i^j(\tau)}{\rho_i} \tag{12}$$

Moving back to $t = 0$, it's noted that investors, devoid of additional data from private signals, are unified under a shared public signal denoted as s_0. The model posits that heightened attention allocation correlates with augmented precision in the aggregate private signal, consequently elevating the expected utility of investors.

An intricate equation delineates the expected utility:

$$\begin{aligned} E_k[U(W_k(t_i))|\mathcal{F}_{t_{i-1}}] &= -e^{\rho_k(W_k(t_{i-1})+w_k(t_{i-1}))}E_k[e^{-\rho_k(p_{t_{i-1}}\theta_k(t_{i-1})+C[\sum_{j=1}^{J} a_k^j(t_i)]\Delta t)}|\mathcal{F}_{t_{i-1}}] \\ &= -e^{\rho_k(W_k(t_{i-1})+w_k(t_{i-1}))}E_k[e^{-\rho_k(p_{t_{i-1}}[\theta_{k,t_{i-1}}-\frac{s_k(t_{i-1})\sum_{j=1}^{J} a_k^j(t_i)}{\rho_k}]+C[\sum_{j=1}^{J} a_k^j(t_i)]\Delta t)}|\mathcal{F}_{t_{i-1}}] \end{aligned} \tag{13}$$

where the second equality is derived by substituting the preceding Equation (12) and integrating over the aggregate private signal.

For the economic intuition of Equation (13), $\rho_k(p_{t_{i-1}}C[\sum_{j=1}^{J} a_k^j(t_i)])$ denotes the cost of information acquisition and $\frac{s_k(t_{i-1})\sum_{j=1}^{J} a_k^j(t_i)}{\rho_k}$ features the expected benefit in terms of utility with precision $\sum_{j=1}^{J} a_k^j(t_i)$. Before solving the problem in $t = 0$, we first involve the risk-neutral conditional expectation based on the Radon-Nikodyn theorem.

$$\bar{E}(x) := E[\frac{U'}{E(U')}x] \tag{14}$$

The subscript in the risk-neutral expectation operator is removed since the transformation gets rid of the component related to a particular individual through adjusting. For the optimal attention level, we differentiate Equation (13) with respect to each $a_k^j(t_i)$, respectively, which gives us

$$C'[\sum_{j=1}^{J} a_k^j(t_i)] = \frac{1}{2\rho_k}\bar{E}_0(x^2 - p_1^2) = \frac{1}{2\rho_k}\bar{E}_0[\bar{Var}_1(x)] \tag{15}$$

In the context of Equation (15), the marginal cost associated with attention allocation escalates concomitantly with the level of attention allocated. This phenomenon is elucidated by the right-hand side of the equation, which establishes a proportionality between the marginal cost at inception and the expected value of $x^2 - p_1^2$, as delineated by the risk-neutral expectation operator. The net risk-free rate, being zero as stipulated by our budget constraint, like Equation (5), renders p_{I-1}^2 equivalent to the expected value of x under the risk-neutral measure. This variance, under the risk-neutral measure, serves as an indicator of the financial market's uncertainty from the perspective of the investors.

The underpinning economic rationale here pertains to the marginal utility derived from the information encapsulated within the aggregate signal. The framework mandates a sequential decision-making process wherein an investor is required to judiciously allocate attention to both signals. This allocation is inherently influenced by the anticipated value of the variance under the risk-neutral measure in subsequent periods.

Also, we figured out the portfolio choice in period t is equal to

$$\theta_k(t) = \theta(p_0, p_{t_1}, \dots, p_t) + \frac{1}{\rho_k}\sum_{j=1}^{J} 2a_k^j(t)s_k(t) \tag{16}$$

The additive separability in demand functions for the risky asset, attributable to private signals, remains consistent with the CARA utility specification and the posited structure of independent, multi-

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dimensional, normally distributed imperfect signals. Integration of these demand functions into the utility function, followed by elementary algebraic manipulations, reveals a homogeneity among investors. This homogeneity arises due to uniformity in risk-aversion coefficients and attention cost functions. Consequently, the differentiation in attention towards private signals and the risk-neutral variance of final payoffs in competitive equilibrium are uniform across investors, displaying orthogonality in relation to the actualization of private signals.

3.3 Continuous-Time Limits

At the crux of our analysis lies the exploration of the variance and price dynamics of the risky asset. In transitioning to the continuous time framework, our focal point is the intricate behavior of these two critical variables. The economic landscape, in this context, is delineated by the sequences of asset prices p_t, investors' strategic attention allocations a_t^*, and the exogenous residual supply dynamics of the risky asset n_t. Utilizing the market-clearing condition as a pivotal tool, we adeptly reduce the dimensionality of these state variables, as substantiated in the literature[11].

The ensuing equation, representative of the market equilibrium, integrates investors' demand over the spectrum of potential states, set against the backdrop of the residual supply of the asset:

$$\int_0^1 \{\theta_t(p_0,\ldots,p_t) + \frac{1}{\rho}\sum_{\tau=0}^{t-\Delta t} 2a_\tau^* s_{i\tau}\}\, di = n_t \tag{17}$$

This equality emerges from the aggregation of individual behaviors, as dictated by the law of large numbers, in tandem with the market-clearing condition. Consequently, we derive the following relation:

$$\frac{\rho}{2a_t^*\Delta t}(\theta_{t+\Delta t} - \theta_t) = x + \frac{\rho}{2a_t^*\Delta t}(n_{t+\Delta t} - n_t) \tag{18}$$

Here, the public signal approximates x, with an accompanying error term conforming to a normal distribution characterized by zero mean and a variance of $\frac{\rho\sigma_{n_t}}{2(a_t^*)^2\sqrt{\Delta t}}$.

Under the risk-neutral measure, p_t exhibits martingale properties. The variance within this framework at time t is a composite function of the expected future variance and the current variance of the expected future values. Incorporating investors' first-order conditions relative to their portfolio choices, we articulate the price dynamics as:

$$p_t = E_t[e^{-\rho\theta_t(p_{t+\Delta t}-p_t)-\frac{\sum_{\tau=0}^{t-\Delta t} 2a_t^*\Delta t}{2}(p_{t+\Delta t}^2-p_t^2)+\rho C(2a_t^*)\Delta t+\omega_t} p_{t+\Delta t}$$

$$\bar{Var}_{t+\Delta t}(x) = \bar{E}_t[e^{-\rho\theta_t(p_{t+\Delta t}-p_t)-\frac{\sum_{\tau=0}^{t-\Delta t} 2a_t^*\Delta t}{2}(p_{t+\Delta t}^2-p_t^2)+\rho C(2a_t^*)\Delta t+\omega_t} p_{t+\Delta t}] + \bar{Var}_t(p_{t+\Delta t}) \tag{19}$$

Applying Taylor expansion to these expressions and transitioning Δt towards zero, we attain the continuous time limit. This culminates in a system of partial differential equations adeptly encapsulating the dynamics of the model.

4. Conclusions

This paper proposes a noisy rational expectation model incorporating endogenous dynamics between information acquisition and attention allocation. The key finding is that investors' attention allocation choices are correlated with the anticipated value of the risky asset's terminal payoff under the risk-neutral measure. In particular, the marginal cost of attention is proportional to the expected variance of the terminal payoff.

The model shows that when facing greater market uncertainty, measured by a higher risk-neutral variance, investors will devote more attention to acquiring and processing relevant information. This helps explain information demand fluctuations in financial markets. The theoretical model could be empirically tested using options-implied volatility indices as a proxy for risk-neutral variance. In an empirical context, the SVIX index approximates the risk-neutral variance of the terminal payoff.

A promising avenue for future research is to relax assumptions made for tractability to explore more complex investor behaviors. For example, allowing for heterogeneous risk preferences and asymmetric

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information may lead to new insights. The attention allocation framework could also be integrated into other asset pricing models. Overall, incorporating an endogenous information acquisition process opens up new possibilities for understanding financial market dynamics.

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