

The Lanchester Mathematical Model Analysis of the Battle of Sarhu

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ABSTRACT. *Battle of Sarhu is a campaign that a small number of forces won most forces in chinese history. the post-Jin dynasty army defeated the Ming dynasty army, which was nearly twice as strong and won the war. This paper analyzes the result of wars through the establishment of Lanchester model, explaining the reason why the post-Jin dynasty army can defeat the Ming dynasty army at a inferiority in the form of visual force data. Analyzing the development trend of the war, and obtaining the conclusion of the importance of tactical use.*

KEYWORDS: *Battle of Sarhu; Lanchester mathematical model*

1. The Background of the Battle of Sarhu

1.1 The Battle of Sarhu

The battle of Sarhu was in February and March of 1619 (forty-seven years in the Ming dynasty and four years in the post-Jin dynasty). Nurhachi defeated the Ming army's four-way offensive in Sarhu (now on the South Bank of Fushun, Liaoning Province) and in the vicinity of Sarhu. It was a strategic decisive battle between the Ming dynasty and the post-Jin dynasty.

1.2 The Operational Policy of the Ming Dynasty

In the Ming dynasty, Yanghao's combat policy was to take Hetuala as the target, divide into joint attacks, attack in four routes and encircle and annihilate the post-Jin dynasty army in one fell swoop.

About 30,000 elite cavalry troops were dispatched from Xuanfu, Datong and Shanxi townships in the Ming dynasty. The four towns of Yansui, Ningxia, Gansu and Guyuan dispatched about 25,000 troops. Sichuan, Guangdong, Shandong, Shaanxi, north direct and south direct dispatched about 20,000 troops. Zhejiang dispatched 4,000 Zhejiang infantry. The number of Tusi soldiers in Yongshun, Baojing and Shizhou ranges from 2,000 to 3,000, with a total of about 7,000 soldiers.

The total number of the Ming dynasty army was about 86,000. With 10,000 allies and 13,000 Korean troops, a total of more than 110,000 people, known as 470,000 people.

The specific deployment of the Ming dynasty forces was as follows: Yang Hao conducted command in Shenyang. Ma Lin led 15,000 people to attack from the north. Du Song led the main force of about 30,000 troops to attack from the west. Li Rubai led 25,000 troops to attack from the southwest. Liu Ting led more than 10,000 troops and plus the Korean army, more than 20,000 people attacked from the south.

1.3 The Operational Policy of the Post-Jin Dynasty

Before the Ming dynasty army set out in four directions, the battle attempt was detected by the post-Jin dynasty army. Knowing the Ming dynasty army's actions, Nurhachi believed that the Ming dynasty army had a long way to go and could not reach it immediately. He thought it was necessary to break down one by one, so he decided to adopt the policy of "no matter how many routes you come, I will only attack one way" to gather force and break down one by one. He concentrated 60,000 troops near the capital to prepare for battle.

2. The Mathematical Model Used in Case Analysis

2.1 The Lanchester Equation

Lanchester is a famous British automotive engineer, fluid mechanic and operational research scientist. He was the first scientist to make a mathematical analysis of the force relations between the opposing parties in the course of battle. He established the Lanchester equation, which is based on a mathematical measure of the concentration effect of troops or firepower, and established a mathematical equation containing this measure to describe and predict the development trend of the battle process. It shows that under certain circumstances and conditions a military with inferior numbers can defeat an army with superior numbers[1].

Lanchester differential equation is a set of differential equations describing the relationship between the forces of both sides in the course of warfare. In 1915, F.W. Lanchester, a British engineer, first published this paper, which was based on the different characteristics of using cold weapons to fight in ancient times and guns to fight in modern times, and under some simplified assumptions, the corresponding differential equations were established, which profoundly revealed the quantitative relation of force change between the two parties during the war[2].

2.2 The Lanchester Square Law

Lanchester direct firing model was established on the basis of modern combat model. At time t , the number of blue combat units or weapons is $m_B(t)$, the loss

coefficient is α_B , the number of red combat units or weapons is $m_R(t)$, the loss coefficient is α_R .

The Lanchester direct firing model is called the square law model, which makes the following assumptions: the two sides have the same type of forces, and the battle is going on continuously. Each unit or weapon of one side is completely exposed to the maximum range of the weapon of the other side. The cumulative damage of the weapon to the target is not considered and the wear rate coefficient contains the meaning of the probability of the target being destroyed under hit condition. The effective firing rate of one side has nothing to do with the military strength level of the other side. The firing of one side's weapons is evenly distributed on the other side's combat units or weapons. Each combat unit on one side knows exactly the location and status of all combat units on the other side[1].

The ordinary differential equations of Lanchester direct firing model are as follows.

$$\begin{aligned}\frac{dm_B}{dt} &= -\alpha_R m_R \\ \frac{dm_R}{dt} &= -\alpha_B m_B\end{aligned}$$

The initial condition is $m_B(0) = m_{B0}$, $m_R(0) = m_{R0}$.

The state solution of Lanchester direct firing model can be obtained from the above equation.

$$\frac{dm_B}{dm_R} = \frac{\alpha_R m_R}{\alpha_B m_B}$$

The following equations are obtained by integrating the above equation.

$$\alpha_B \int_{m_{B0}}^{m_B} m_B dm_B = \alpha_R \int_{m_{R0}}^{m_R} m_R dm_R$$

The following equations are obtained after sorting out.

$$\alpha_B(m_{B0}^2 - m_B^2) = \alpha_R(m_{R0}^2 - m_R^2)$$

The above equation is the state solution of Lanchester direct firing model, which obeys the square law. This is the state solution that the two sides should satisfy in the course of battle.

3. The Establishment of Lanchester Combat Model

According to the combat characteristics of the Ming army and the post-Jin dynasty army, Lanchester square ratio can intuitively reflect the combat effectiveness comparison between the two sides. Therefore, the following mathematical models are established.

$$\alpha_B(m_{B0}^2 - m_B^2) = \alpha_R(m_{R0}^2 - m_R^2)$$

Among them, α_B indicates the killing rate of the army of the post-Jin dynasty to the army of Ming dynasty, α_R indicates the killing rate of the army of Ming dynasty to the army of the post-Jin dynasty, m_{B0} indicates the fighting capacity of the army of the post-Jin dynasty at the beginning of the battle, m_{R0} indicates the fighting capacity of the army of the Ming dynasty at the beginning of the battle, m_B indicates the remaining fighting capacity of the post-Jin dynasty army after its victory, m_R indicates the remaining fighting capacity of the post-Jin dynasty army after its defeat.

Assuming that the killing rates of both sides are 1 and fighting effectiveness is measured in 10,000 people and then the data of each battle are analyzed by the methods and the use of forces adopted by both sides.

4. The Analysis of the Battle of Sarhu by Lanchester Square Law

The fact of ancient warfare is that when the loss of a party exceeds a certain amount (which is usually not high), it often collapses because of low morale, and then becomes "pursue and attack". In this case, the battle turns into massacre, and the killing rate of the disintegrated army is about 0. So let's assume that both sides will collapse when they lose half of their own forces.

4.1 The Main Forces of the Ming Dynasty Army were Annihilated

On February 29, 1619, the post-Jin dynasty army discovered that the Ming dynasty vanguard troops moved northward from Kuandian, and Du Song led the Ming dynasty army main force to move eastward from Fushu, but the progress was too rapid and isolated. After receiving the report, Nurhachi decided to hamper the Ming dynasty army vanguard with 500 troops originally stationed in Hetualan and then take advantage of the slow progress of other Ming dynasty army routes to concentrate eight-flag forces to attack Du Song's army.

On the first day of March, 1619, Du Song's army rushed into the Sarhu, divided into two parts. The main force was stationed near the Sarhu, and he led 10,000 troops to attack the Jilin cliff. Nurhachi found that Du Song's army had gone deep alone. He first sent troops to reinforce the Jilin cliff, and then personally led 45,000 people to attack the main army of Sarhu. According to the formula of Lanchester square law, the calculation is as follows.

$$m_{B1}^2 - 1^2 = 4.5^2 - 2^2$$

$$m_{B1} = 4.15$$

After the army of the post-Jin dynasty annihilated the army of the Ming dynasty near Sarhu, it immediately supported the Jilin cliff area. The calculation is as follows.

$$m_{B2}^2 - 0.5^2 = 4.15^2 - 1^2$$

$$m_{B2}=4.06$$

According to the model analysis, after the battle, the post-Jin dynasty army annihilated 15,000 troops of the Ming dynasty, and the post-Jin dynasty army used 500 troops to hamper Liu Ting's army of the Ming dynasty. Liu Ting's force totaled 30,000 people, so all the 500 people of the post-Jin dynasty were annihilated. Therefore, after the post-Jin dynasty army annihilated the main force of the Ming army stationed in the Sarhu area, the total strength of the post-Jin dynasty army was 55,600 people.

4.2 The Defeat of the Ming Dynasty Army in the North

On the night of March 1st, 1619, Ma Lin led the army of the North Road of the Ming Dynasty to go to Shangjianya (in the northeastern direction of Sarhu). After he knew that Du Song's army was defeated, he dared not advance. He stationed his army in three local defenses. After annihilating Du Song's army, Nurhachi led the main army northward to attack Ma Lin's army in the direction of Shangjianya. The calculation is as follows.

$$m_{B3}^2 - 0.75^2 = 5.56^2 - 1.5^2$$

$$m_{B3} = 5.41$$

Therefore, according to the model analysis, after the defeat of the northern Ming dynasty army, the remaining strength of the post-Jin dynasty army was 54100 people. At the same time, the army of the Ming dynasty was annihilated to 225,00 troops.

4.3 The Defeat of the Ming Dynasty Army in the East

The Eastern army led by Liu Ting was unable to reach Hetuala on schedule because of its rugged mountain roads and difficult movements. He did not know that the west army and the north army had been defeated, but he still went north as planned. After the defeat of Ma Lin's army, Nurhachi immediately moved to attack Liu Ting's army. The calculation is as follows.

$$m_{B4}^2 - 1.5^2 = 5.41^2 - 3^2$$

$$m_{B4} = 4.75$$

Therefore, according to the model analysis, after the defeat of the northern Ming dynasty army, the remaining strength of the post-Jin dynasty army was 47500 people. At this time, the army of the Ming dynasty was annihilated to 375,00 troops.

4.4 The Defeat of the Ming Dynasty Army in the South

Yang Hao led the troops to guard Shenyang, but he did not make any response to the third route Ming dynasty army. He waited until the defeat of Du Song and Ma Lin, and on the fifth day of March, he hastily summoned Li Rubai's army back to

camp. Li Rubai's army was slow to move and only arrived at Hulanang. When receiving the withdrawal order, the Ming dynasty army was found by the post-Jin sentinel. The post-Jin sentinel sent a shock signal on the mountain and shouted loudly. Li Rubai thought it was the post-Jin's main force that launched an attack, fled in panic, killing and injuring more than 1,000 people.

Therefore, after the defeat of the Ming dynasty army on the southern line, the Ming dynasty army lost a total of 38500 people.

5. The Result of the Battle of Sarhu

The battle of Sarhu ended with the defeat of the Ming dynasty army and the victory of the post-Jin dynasty army. In this campaign, the post-Jin dynasty army used the principle of concentrating its forces and breaking through each other in its operational command. Within five days, it defeated the third route Ming dynasty army, annihilated about 50,000 troops of the Ming dynasty and captured a large amount of military materials. This campaign ended in a great defeat of the Ming dynasty army.

According to the data, the main forces of the Ming dynasty led by Du Song were totally wiped out, most of the troops of the Ming dynasty in the north were wiped out, and some of the troops of the Ming dynasty in the east were wiped out, so the army of the Ming dynasty was wiped out about 50,000 people after the war. However, in the battle model, we assume that when the military strength of the Ming dynasty lost more than half, the army of the Ming dynasty collapsed because of low morale, so only half of the army of the Ming dynasty was annihilated in each battle. So we calculated that the loss of the army of the Ming dynasty was 38,500 people. However, after the defeat of the main force of the Ming dynasty army and the army of the north road, the post-Jin dynasty army wiped out all the remaining forces of the Ming dynasty army, which resulted in the difference between the calculated number of enemies and the actual number of deaths, but did not affect the actual results of the final victory of the post-Jin dynasty army.

6. Conclusion

The results show that each of these local battles in the Sarhu campaign is dominated by the troops of the post-Jin dynasty, which leads to that the winners of each battle are the post-Jin dynasty army, so the final winner of the battle is also the post-Jin dynasty army. At first, the army of the Ming dynasty was twice as strong as that of the post-Jin dynasty army, but the army of the Ming dynasty adopted the tactics of dividing into four directions to attack and annihilate the post-Jin dynasty army. As a result, the army of the Ming dynasty in all directions was significantly less than that of the post-Jin dynasty army. The army of the Ming dynasty had 15,000 troops in the north, 30,000 troops in the west, 25,000 troops in the South and 30,000 troops in the east. However, the post-Jin dynasty army was aware of the Ming dynasty army's combat intentions in advance, and 60,000 troops defeated the

Ming dynasty army in all directions. According to lanchester square law, the battle effectiveness of the post-Jin dynasty army has been more than four times that of the Ming dynasty army, so the post-Jin dynasty army can wipe out the Ming dynasty army with very little loss.

Of course, this is only a simplified mathematical model, ignoring some factors that are difficult to quantify, such as weather, geography, human and political factors, which also have a decisive impact on war. In fact, from a scientific point of view, research results are only valid for research models, but research always starts with basic models. Therefore, through the lanchester battle model, we can speculate on the basic trend of a battle and analyze the battle action in order to make appropriate decisions.

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