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Row-Column Block Designs with Blocks of Size Three

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Abstract: *In this study, we construct row-column designs with three elements per cell and with the following properties: (1) the blocks formed by the cells form a balanced incomplete block design; (2) the rows are regular, in the sense that each element occurs the same number of times in each row; and (3) the columns are near-regular, meaning there is an integer x such that each element occurs either x or x + 1 times. This generalizes the work done in (Seema, Cini and Eldho, 2016) in which such row-column designs are constructed with block size 2. The constructions make use of known solutions to Heffter's Difference Problem.*

Keywords: *Row-Column Designs, Heffter Difference Sets, Balanced Incomplete Block Design (BIBD)*

1. Introduction

For further definitions of combinatorial structures, we refer the reader to [1]. For further information on the application of combinatorial designs to experimental design, we refer the reader to [2].

A row-column design, in its most general sense, is any rearrangement of the blocks of a combinatorial design into a rectangular array. Thus, a row-column design admits two partitions of blocks, given by the rows and columns. Most of the row-column designs developed in the literature have one unit corresponding to the intersection of row and column. Examples of such row-column designs are: Latin squares, Youden squares and generalized Youden designs [3]. Row-column designs with more than one unit per cell include: semi-Latin squares [4] and Trojan squares [5]. Such designs are used when the number of elements is substantially large with a limited number of replicates [6].

In this paper, we consider the an $m \times n$ row-column block design is any rearrangement of the blocks of a Balanced Incomplete Block Design BIBD(v, k, λ) into an $m \times n$ array. We say that a row or column is regular, if each element occurs the same number of times in that row or column. Regularity is desirable in the context of experimental design because this ensures that a row or column effect is not "confounded" with the effect of an individual treatment (see [2] for more information). We say that a row or column is near-regular if it is not regular but there is an integer x such that every entry occurs either x or x + 1 times.

We assume the set of treatments in BIBD (v, k, λ) is given by Z_v . A row-column block design is in said to be row-cyclic if whenever B is a block in cell (i, j),

$$B' = \{b + 1 \pmod v \mid b \in B\} \tag{1}$$

Is the block in cell (i, j + 1 (mod n)). We define column-cyclic similarly. The above definitions are demonstrated in the example given in Table 1.

Table 1: A row-cyclic 5 × 7 array row-column block design of index 3, where each row is regular and each column is near-regular.

1, 2, 3	2, 3, 4	3, 4, 5	4, 5, 6	5, 6, 0	6, 0, 1	0, 1, 2
4, 5, 0	5, 6, 1	6, 0, 2	0, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6
5, 6, 2	6, 0, 3	0, 1, 4	1, 2, 5	2, 3, 6	3, 4, 0	4, 5, 1
6, 0, 4	0, 1, 5	1, 2, 6	2, 3, 0	3, 4, 1	4, 5, 2	5, 6, 3
6, 1, 3	0, 2, 4	1, 3, 5	2, 4, 6	3, 5, 0	4, 6, 1	5, 0, 2

The problem of constructing row-column block designs with index 2 was considered in [9]:

Theorem 1.1. [9]

For any t, there exists:

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(a) a row-cyclic t × (2t + 1) row column block design of index 2, where each row is regular and each column is near-regular.

(b) a row-cyclic 2t × 2t row column block design of index 2, where each row is near-regular and each column is regular.

The initial columns for this construction are shown in Tables 2 and 3.

Table 2: Odd Case for Theorem [9]

1, 2t + 1
2, 2t
3, 2t − 1
:
t − 1, t + 3
t, t + 2

Table 3: Even Case for Theorem [9]

1, 2
2t, 2
2, 2t − 1
:
t + 2, t
t, t + 1

The construction uses the classic Walecki cyclic one-factorization of the complete graph[10].

In this paper, we generalize the previous theorem to index 3. The main theorem of this paper is as follows.

Theorem 1.2. For any v ≡ 1 mod 6 there exists a row-cyclic row-column block design with (v − 1)/6 rows, v columns and index 3 such that each row is regular and each column is near-regular.

Our proof of the above theorem in Section 4 will make use of Heffter difference sets, see Section 3. We also find more specific results for v = 7 and v = 13 in Section 2. These are summarized in Theorem 3.3.

For future work, we make the following comment. For experimental design applications, it may also be desirable that block repetition is minimized. In fact, this can be achieved nicely in the case of the 5 × 7 row column design in Table 17, where we observe that each of the possible 35 triples from Z7 occurs exactly once. An open problem to consider is the following: For each v ≡ 1 (mod 6), does there exist a (v − 1)(v − 2)/6 × v row-column block design of index 3, where each row is regular and each column is near-regular such that each triple from Zv occurs exactly once?

2. Row-column block designs for 7 and 13 columns

In this section we give specific results for a small number of columns. Firstly we give the following theorem, then it can be easily proved by the table enumeration method. Among them, tables with 7 columns take m values from 1 to 7, and tables with 13 columns take m values 2 or 4.

Theorem 2.1. For any m ≥ 1, there exists a row-cyclic m × 7 row-column block design of index 3 where each row is regular and each column is regular (if m is divisible by 7) or near-regular (if m is not divisible by 7).

Proof. We first show the above theorem is true for the cases 1 ≤ m ≤ 7.

See Table 4- Table 10 :

Table 4 : A row-cyclic 1 × 7 row-column block design of index 3, where each row is regular and each column is near-regular.

0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2

Table 5: A row-cyclic 2 × 7 array row-column block design of index 3, where each row is regular and each column is near-regular.

0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2
5, 6, 1	6, 0, 2	0, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0

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Table 6: A row-cyclic 3 × 7 array row-column block design of index 3, where each row is regular and each column is near-regular.

0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2
5, 6, 1	6, 0, 2	0, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0
4, 0, 6	5, 1, 0	6, 2, 1	0, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5

Table 7: A row-cyclic 4 × 7 array row-column block design of index 3, where each row is regular and each column is near-regular.

0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2
5, 6, 1	6, 0, 2	0, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0
4, 0, 6	5, 1, 0	6, 2, 1	0, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5
1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0	5, 6, 1	6, 0, 4	0, 1, 3

Table 8: A row-cyclic 5 × 7 row-column block design of index 3, where each row is regular and each column is near-regular.

0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2
5, 6, 1	6, 0, 2	0, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0
4, 0, 6	5, 1, 0	6, 2, 1	0, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5
1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0	5, 6, 1	6, 0, 2	0, 1, 3
3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2	0, 2, 3	1, 3, 4	2, 4, 5

Table 9: A row-cyclic 6 × 7 array row-column block design of index 3, where each row is regular and each column is near-regular.

0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2
5, 6, 1	6, 0, 2	0, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0
4, 0, 6	5, 1, 0	6, 2, 1	0, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5
1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0	5, 6, 1	6, 0, 2	0, 1, 3
3, 5, 7	4, 6, 0	5, 0, 1	6, 1, 2	0, 2, 3	1, 3, 4	2, 4, 5
0, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 0	5, 6, 1	6, 0, 2

Table 10: A row-cyclic and column-cyclic 7 × 7 row-column block design of index 3, where each row is regular and each column is also regular.

0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2
1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2	0, 2, 3
2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2	0, 2, 3	1, 3, 4
3, 5, 6	4, 6, 0	5, 0, 1	6, 1, 2	0, 2, 3	1, 3, 4	2, 4, 5
4, 6, 0	5, 0, 1	6, 1, 2	0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6
5, 0, 1	6, 1, 2	0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0
6, 1, 2	0, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 0	5, 0, 1

See table 11- Table 12 :

If $m > 7$, let $m = 7a + m'$. A solution is then formed by adjoining a copies of the solution for 7 rows with one copy of the solution for m' rows.

Next, Tables 11 and 12 show a row cycle with index 3 and a column cycle-column block design.

Table 11: A row-cyclic and column-cyclic 2 × 13 row-column block design of index 3, where each row is regular and each column is near-regular.

0, 1, 4	1, 2, 5	2, 3, 6	3, 4, 7	4, 5, 8	5, 6, 9	6, 7, 10	7, 8, 11	8, 9, 12	9, 10, 0	10, 11, 1	11, 12, 2	12, 0, 3
3, 5, 10	4, 6, 11	5, 7, 12	6, 8, 0	7, 9, 1	8, 10, 2	9, 11, 3	10, 12, 4	11, 0, 5	12, 1, 6	0, 2, 7	1, 3, 8	2, 4, 9

Table 12: A row-cyclic and column-cyclic 4 × 13 row-column block design of index 3, where each row is regular and each column is near-regular.

0,1,4	1,2,5	2,3,6	3,4,7	4,5,8	5,6,9	6,7,10	7, 8, 11	8, 9, 12	9,10,0	10,11,1	11,12,2	12, 0, 3
3,5,10	4,6,11	5,7,12	6,8,0	7,9,1	8,10,2	9,11,3	10, 12, 4	11, 0, 5	12, 1, 6	0, 2, 7	1, 3, 8	2, 4, 9
2,11,12	8,9,12	9,10,0	10,11,1	11,12,2	12,0,3	0,1, 4	1, 2, 5	2, 3, 6	3, 4, 7	4, 5, 8	5, 6, 9	6, 7, 10
6,7,9	3,7,10	4,8,11	5,9,12	6,10,0	7,11,1	8, 12, 2	9, 0, 3	10, 1, 4	11, 2, 5	12, 3, 6	0, 4, 7	1, 5, 8

According to the above table enumeration method, as long as we get one feasible row-column block design, and the feasibility designs proved. Of course we can get more feasible row-column block designs through the change of elements of each cell. At the same time, we get the following lemma for n × 13 row-column block design of index 3.

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Lemma 2.2. If n ∈ {2, 4}, there exists a row-cyclic n × 13 row-column block design of index 3 where each row is regular and each column is regular (if n is divisible by 13) or near-regular (if n is not divisible by 13).

3. Row-column block designs from solutions to Heffter's Difference Problem

In this section we discuss the relation between row-column block designs and the solutions to Heffter's Difference Problem. Heffter's Difference Problem is a problem that has been concerned in mathematics for a long time, and he pays attention to the discussion of the relationship between graph theory and combinatorial mathematics, and uses the congruence modulo in number theory to solve the problem.

A difference triple {a, b, c} is a set of three different elements from {1, 2, . . . , v − 1}, whose sum modulo v equals zero (a + b+ c = 0 (mod v)) or for which one element mod v equals the sum of the other two (a + b = c (mod v)).

Heffter proposed the following problems:

(1) First Difference Problem of Heffter:

Let v = 6m + 1. Is there a partition of the set {1, 2, . . . , 3m} in difference triples?

(2) Second Difference Problem of Heffter:

Let v = 6m + 3. Is there a partition of the set {1, 2, . . . , 2m} ∪ {2m + 2, 2m + 3, . . . , 3m + 1} in difference triples?

For example, {{1, 5, 6}, {2, 8, 10}, {3, 4, 7}} is a solution to Heffter's first difference problem for v = 19. Solutions to Heffter difference problems are known for all cases except v = 9:

Theorem 3.1. (Peltesohn, [8]) There exists a solution to Heffter's first and second difference problem for each m ≥ 1 with the exception of the case v = 9.

Any solution of the first difference problem of Heffter also gives a construction of a BIBD(v, 3, 1) (also known as a Steiner triple system) which has the extra property of being cyclically generated.

We explain this process as follows. Given two elements a and b of Zv , the difference of {a, b}, denoted by d(a, b), is defined to be the minimum value of a − b (mod v) and b − a (mod v). Given a subset T of Zv , we define d(T) to be the set of differences that arise from S. That is,

$$d(T) = \{d(a, b) \mid \{a, b\} \subseteq T\} \tag{2}$$

In turn, given s, a set of subsets of Zv , we define

$$\Delta(S) = \bigcup_{T \in S} d(T) \tag{3}$$

Now, for any difference triple B, we can easily construct a set T such that d(T) = B. If B has the form {a, b, a + b}, then let T = {0, a, a + b}. Otherwise, if B has the form {a, b, c} where a + b + c = 0 (mod v), then similarly let T = {0, a, a + b}. By cycling such a set of blocks modulo v, we create a BIBD(v, 3, 1).

For example, consider the solution B={{1,5,6},{2,8,10},{3,4,7}} to Heffter's first difference problem for v=19, as given above, following the construction method described earlier, we can obtain the corresponding set T={{0,1,6},{0,2,10},{0,3,7}}. Notice that the differences generated by T cover all integers from 1 to 9, i.e., Δ(T)={1,2,…,9}. Therefore, this construction successfully yields a BIBD with the desired properties.

Table 13 is a BIBD(19, 3, 1).

Table 13: A cyclic BIBD(19, 3, 1)

0, 1, 6	1, 2, 7	2, 3, 8	···	16, 17, 3	17, 18, 4	18, 0, 5
0, 2, 10	1, 3, 11	2, 4, 12	···	16, 18, 7	17, 0, 8	18, 1, 9
0, 3, 7	1, 4, 8	2, 5, 9	···	16, 0, 4	17, 1, 5	18, 2, 6

Now we are ready to see the connection between solutions to Heffter's first difference problem and row-column block designs with block size 3.

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Theorem 3.2. Suppose there exists a set S of (v − 1)/6 triples from Zv such that:

(1) Δ(S) is the set {1, 2, . . . , (v − 1)/2};

(2) The set S is near-regular.

Then, there exists a row-cyclic row-column block design with (v − 1)/6 rows, v columns and index 3 such that each row is regular and each column is near-regular.

For example, for v = 19, we can replace T from above with a set T ′ such that Δ(T) = Δ(T ′) but T ′ is near- regular. These triples form the first column of the 3 × 19 row-cyclic row-column block design given in Table 15, which is also row-regular and column near-regular.

We prove the following theorem in the next section. As demonstrated in the above example, together with the above theorem, this implies Theorem 1.2.

Theorem 3.3. For any v ≡ 1 mod 6 there exists a set S of (v − 1)/6 triples such that:

The triples from S form a solution to Heffter's first difference problem;

The set S is near-regular.

We also prove an equivalent version of the above theorem in the context of Heffter's second difference problem in Section 5. In this case, the row-column designs created are not row-column block designs; in particular pairs of treatments of difference v/3 are never included.

Theorem 3.4. For any v ≡ 3 mod 6 there exists a set S of (v − 3)/6 triples such that:

The triples from S form a solution to Heffter's second difference problem;

The set S is near-regular.

We will make use of the following solutions to Heffter's difference set problem (originally in [8], see also [7]) in Sections 4 and Section 5.

The following Lemmas are got by Theorem 3.3 and Theorem 3.4 with different congruence values of v. (v ≡ 1,3,7,9,13,15 (mod 18)) And the following relation of r and s is directly from Heffter's difference set problem [8].

Lemma 3.5. [8] Let v ≡ 1 (mod 18) and v ≥ 19. Then the following is a solution to Heffter's first difference problem, where v = 18s + 1.

$$\begin{gathered}\{\{3r+1,\ 4s-r+1,\ 4s+2r+2\}\mid 0\le r\le s-1\};\\ \{\{3r+2,\ 8s-r,\ 8s+2r+2\}\mid 0\le r\le s-1\};\\ \{\{3r+3,\ 6s-2r-1,\ 6s+r+2\}\mid 0\le r\le s-2\};\\ \{\{3s,\ 3s+1,\ 6s+1\}\}.\end{gathered}\tag{4}$$

Lemma 3.6. [8] Let v ≡ 3 (mod 18) and v ≥ 21. Then the following is a solution to Heffter's second difference problem, where v = 18s + 3.

$$\begin{gathered}\{\{3r+1,\ 8s-r+1,\ 8s+2r+2\}\mid 0\le r\le s-1\};\\ \{\{3r+2,\ 4s-r,\ 4s+2r+2\}\mid 0\le r\le s-1\};\\ \{\{3r+3,\ 6s-2r-1,\ 6s+r+2\}\mid 0\le r\le s-1\}.\end{gathered}\tag{5}$$

Lemma 3.7. [8] Let v ≡ 7 (mod 18) and v ≥ 25. Then the following is a solution to Heffter's first difference problem, where v = 18s + 7.

$$\begin{gathered}\{\{3r+1,\ 8s-r+3,\ 8s+2r+4\}\mid 0\le r\le s-1\};\\ \{\{3r+2,\ 6s-2r+1,\ 6s+r+3\}\mid 0\le r\le s-1\};\\ \{\{3r+3,\ 4s-r+1,\ 4s+2r+4\}\mid 0\le r\le s-1\};\\ \{\{3s+1,\ 4s+2,\ 7s+3\}\}.\end{gathered}\tag{6}$$

Lemma 3.8. [8] Let v ≡ 9 (mod 18) and v ≥ 81. Then the following is a solution to Heffter's second difference problem, where v = 18s + 9.

$$\{\{3r+1,\ 4s-r+3,\ 4s+2r+4\}\mid 0\le r\le s\};$$

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$$\{\{3r + 2, 8s - r + 2, 8s + 2r + 4\} \mid 0 \le r \le s - 2\};$$
$$\{\{3r + 3, 6s - 2r + 1, 6s + r + 4\} \mid 0 \le r \le s - 2\}; \quad (7)$$
$$\{\{2, 8s + 3, 8s + 5\}, \{3, 8s + 1, 8s + 4\}, \{5, 8s + 2, 8s + 7\},$$
$$\{3s - 1, 3s + 2, 6s + 1\}, \{3s, 7s + 3, 8s + 6\}\}.$$

Lemma 3.9. [8] Let $v \equiv 13 \pmod{18}$ and $v \ge 31$. Then the following is a solution to Heffter's first difference problem, where $v = 18s + 13$.

$$\{\{3r + 1, 4s - r + 3, 4s + 2r + 4\} \mid 0 \le r \le s\};$$
$$\{\{3r + 2, 6s - 2r + 3, 6s + r + 5\} \mid 0 \le r \le s - 1\}; \quad (8)$$
$$\{\{3r + 3, 8s - r + 5, 8s + 2r + 8\} \mid 0 \le r \le s - 1\};$$
$$\{\{3s + 2, 7s + 5, 8s + 6\}\}.$$

Lemma 3.10. [8] Let $v \equiv 15 \pmod{18}$ and $v \ge 33$. Then the following is a solution to Heffter's second difference problem, where $v = 18s + 15$.

$$\{\{3r + 1, 4s - r + 3, 4s + 2r + 4\} \mid 0 \le r \le s\};$$
$$\{\{3r + 2, 8s - r + 6, 8s + 2r + 8\} \mid 0 \le r \le s\}; \quad (9)$$
$$\{\{3r + 3, 6s - 2r + 3, 6s + r + 6\} \mid 0 \le r \le s - 1\}.$$

4. Heffter's first difference problem

In this section according to the Theorem 3.3 and the corresponding Lemma given in the previous section, it can be divided into Heffter's first difference problem and Heffter's second difference problem with the change of v. Then Heffter's first difference problem is summarized together with congruence value of $v \equiv 1,7,13 \pmod{18}$ as follows. We use every congruence values of v to substitute into the theorem in section 3 to get new Lemmas, at the same time we use the values of r and s to make a table and prove it.

Here we give a solution to Theorem 3.3, splitting into the following cases:

Case 1: $v \equiv 1 \pmod{18}$.

Case 2: $v \equiv 7 \pmod{18}$.

Case 3: $v \equiv 13 \pmod{18}$.

Lemma 4.1. Let $v \equiv 1 \pmod{18}$. Then there exists a set S of $(v - 1)/6$ triples such that:

(1) $\Delta(S)$ is the set $\{1, 2, \ldots, (v - 1)/2\}$;

(2) The set S is near-regular.

Proof. Let $v = 18s + 1$. First consider the case $s = 1$. Let $S = \{\{1, 6, 7\}, \{0, 2, 9\}, \{3, 6, 10\}\}$. Then $\Delta(S) = \{1, 5, 6\} \cup \{2, 8, 9\} \cup \{3, 4, 7\}$.

Otherwise, $s \ge 2$. Let

$$S_1 = \{\{1 - r, 2 + 2r, 4s + 3 + r\} \mid 0 \le r \le s - 1\};$$
$$S_2 = \{\{6s + 2 - r, 6s + 4 + 2r, 14s + 4 + r\} \mid 0 \le r \le s - 1\}; \quad (10)$$
$$S_3 = \{\{2s - 1 - 2r, 2s + 2 + r, 8s + 1 - r\} \mid 0 \le r \le s - 2\};$$
$$S_4 = \{\{3s + 3, 6s + 3, 9s + 4\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied.

This can be more clearly seen in Table 14.

Then, observe that:

$$\Delta(S_1) = \{\{3r + 1, 4s - r + 1, 4s + 2r + 2\} \mid 0 \le r \le s - 1\};$$

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$$\Delta(S_2) = \{\{3r + 2, 8s - r, 8s + 2r + 2\} \mid 0 \le r \le s - 1\}; \quad (11)$$

$$\Delta(S_3) = \{\{3r + 3, 6s - 2r - 1, 6s + r + 2\} \mid 0 \le r \le s - 2\};$$

$$\Delta(S_4) = \{\{3s, 3s + 1, 6s + 1\}\}.$$

From Lemma 3.5, Condition (1) is also satisfied.

Table 14: General solution for v = 18s + 1.

$S_1: r = 0$	$1, 2, 4s + 3$
$r = 1$	$0, 4, 4s + 4$
…	…
$r = s - 1$	$-s + 2, 2s, 5s + 2$
$S_2: r = 0$	$6s + 2, 6s + 4, 14s + 4$
$r = 1$	$6s + 1, 6s + 6, 14s + 5$
…	…
$r = s - 2$	$5s + 4, 8s, 15s + 2$
$r = s - 1$	$5s + 3, 8s + 2, 15s + 3$
$S_3: r = 0$	$2s - 1, 2s + 2, 8s + 1$
…	…
$r = s - 3$	$5, 3s - 1, 7s + 4$
$r = s - 2$	$3, 3s, 7s + 3$
$3s, 3s + 1, 6s + 1$	$3s + 3, 6s + 3, 9s + 4$

Lemma 4.2. Let v ≡ 7 (mod 18). Then there exists a set S of (v − 1)/6 triples such that:

(1) Δ(S) is the set {1, 2, . . . , (v − 1)/2};

(2) The set S is near-regular.

Proof. Let v = 18s + 7. First consider the case s = 1. Let S = {{1, 2, 13}, {3, 5, 12}, {6, 9, 14}, {0, 4, 10}}. Then

Δ(S) = {1, 11, 12} ∪ {2, 7, 9} ∪ {3, 5, 8} ∪ {4, 6, 10}.

Otherwise, s ≥ 2. Let

$$S_1 = \{\{1 - r, 2 + 2r, 8s + 5 + r\} \mid 0 \le r \le s - 1\};$$

$$S_2 = \{\{4s + 1 - 2r, 4s + 3 + r, 10s + 4 - r\} \mid 0 \le r \le s - 1\}; \quad (12)$$

$$S_3 = \{\{11s + 4 - r, 11s + 7 + 2r, 16s + 3 + r\} \mid 0 \le r \le s - 1\};$$

$$S_4 = \{\{4s + 2, 7s + 3, 12s\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied.

This can be more clearly seen in Table 15.

Table 15: General solution of v = 18s + 7

$S_1: r = 0$	$1, 2, 8s + 5$
$r = 1$	$0, 4, 8s + 6$
…	…
$r = s - 1$	$-s + 2, 2s, 9s + 4$
$S_2: r = 0$	$4s + 1, 4s + 3, 10s + 4$
$r = 1$	$4s - 1, 4s + 4, 10s + 3$
…	…
$r = s - 2$	$2s + 1, 5s + 1, 9s + 6$
$r = s - 1$	$2s + 3, 5s + 2, 9s + 5$
$S_3: r = 0$	$11s + 4, 11s + 7, 16s + 3$
…	…
$r = s - 2$	$10s + 6, 12s + 8, 16s + 6$
$r = s - 1$	$10s + 5, 12s + 10, 16s + 7$
$3s + 1, 4s + 2, 7s + 3$	$4s + 2, 7s + 3, 12s$

Then, observe that:

$$\Delta(S_1) = \{\{3r + 1, 8s - r + 3, 8s + 2r + 4\} \mid 0 \le r \le s - 1\};$$

$$\Delta(S_2) = \{\{3r + 2, 6s - 2r + 1, 6s + r + 3\} \mid 0 \le r \le s - 1\}; \quad (13)$$

$$\Delta(S_3) = \{\{3r + 3, 4s - r + 1, 4s + 2r + 4\} \mid 0 \le r \le s - 1\};$$

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$$\Delta(S_4) = \{\{3s + 1, 4s + 2, 7s + 3\}\}.$$

From Lemma 3.7 , condition (1) is also satisfied.

Lemma 4.3. Let $v \equiv 13 \pmod{18}$. Then there exists a set S of $(v - 1)/6$ triples such that:

(1) $\Delta(S)$ is the set $\{1, 2, \ldots, (v - 1)/2\}$;

(2) The set S is near-regular.

Proof. Let $v = 18s+13$. First consider the case $s = 1$. Let $S = \{1, 8, 9\}, \{0, 4, 10\}, \{3, 5, 14\}, \{6, 9, 22\}, \{7, 12, 24\}$.

Then $\Delta(S) = \{1, 7, 8\} \cup \{4, 6, 10\} \cup \{2, 9, 11\} \cup \{3, 13, 15\} \cup \{5, 12, 14\}$.

Otherwise, $s \geq 2$. Let

$$S_1 = \{\{1 - r, 2 + 2r, 4s + 5 + r\} \mid 0 \leq r \leq s\};$$
$$S_2 = \{\{10s + 8 - 2r, 10s + 10 + r, 16s + 13 - r\} \mid 0 \leq r \leq s - 1\}; \quad (14)$$
$$S_3 = \{\{6s + 5 - r, 6s + 8 + 2r, 14s + 13 + r\} \mid 0 \leq r \leq s - 1\};$$
$$S_4 = \{\{3s + 4, 5s + 6, 14s + 6\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied. This can be more clearly seen in Table 16.

Table 16: General solution of v = 18s + 13

$S_1 : r = 0$	$1, 2, 4s+5$
$r = 1$	$0, 4, 4s+6$
…	…
$r = s-1$	$-s+1, 2s+2, 5s+5$
$S_2 : r = 0$	$10s+8, 10s+10, 16s+13$
$r = 1$	$10s+6, 10s+11, 16s+12$
…	…
$r = s-2$	$8s+12, 11s+8, 15s+15$
$r = s-1$	$8s+10, 11s+9, 15s+14$
$S_3 : r = 0$	$6s+5, 6s+8, 14s+13$
…	…
$r = s-2$	$5s+7, 8s+4, 15s+11$
$r = s-1$	$5s+6, 8s+6, 15s+12$
$3s+2, 7s+5, 8s+6$	$3s+4, 5s+6, 14s+6$

Then, observe that:

$$\Delta(S_1) = \{\{3r + 1, 4s - r + 3, 4s + 2r + 4\} \mid 0 \leq r \leq s\};$$
$$\Delta(S_2) = \{\{3r + 2, 6s - 2r + 3, 6s + r + 5\} \mid 0 \leq r \leq s - 1\}; \quad (15)$$
$$\Delta(S_3) = \{\{3r + 3, 8s - r + 5, 8s + 2r + 8\} \mid 0 \leq r \leq s - 1\};$$
$$\Delta(S_4) = \{\{3s + 2, 7s + 5, 8s + 6\}\}.$$

From Lemma 3.9, Condition (1) is also satisfied.

5. Heffter's second difference problem

In this section according to the Theorem 3.4 and the corresponding Lemma given in the previous section, it can be divided into Heffter's first difference problem and Heffter's second difference problem with the change of v. Then Heffter's second difference problem is summarized together with congruence value of $v \equiv 3,9,15 \pmod{18}$ as follows. We use every congruence values of v to substitute into the theorem in section 3 to get new Lemmas, at the same time we use the values of r and s to make a table and prove it.

Here we give a solution to Theorem 3.4, splitting into the following cases:

Case 4: $v \equiv 3 \pmod{18}$.

Case 5: $v \equiv 9 \pmod{18}$.

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Case 6: v ≡ 15 (mod 18).

Lemma 5.1. Let v ≡ 3 (mod 18). Then there exists a set S of (v − 3)/6 triples such that:

(1) Δ(S) is the set {1, 2, . . . , (v − 1)/2};

(2) The set S is near-regular.

Proof. Let v = 18s + 3. First consider the case s = 1.

Let S = {{1, 2, 11}, {4, 6, 10}, {0, 3, 8}, {5, 12, 19}}.

Then Δ(S) = {1, 9, 10} ∪ {2, 4, 6} ∪ {3, 5, 8} ∪ {7}.

Otherwise, s ≥ 2. Let

$$S_1 = \{\{1 - r, 2 + 2r, 8s + 3 + r\} \mid 0 \le r \le s - 1\};$$
$$S_2 = \{\{5s + 1 - r, 5s + 3 + 2r, 9s + 3 + r\} \mid 0 \le r \le s - 1\}; \quad (16)$$
$$S_3 = \{\{7s - 2r, 7s + 3 + r, 13s + 2 - r\} \mid 0 \le r \le s - 1\};$$
$$S_4 = \{\{8s + 3, 14s + 4, 20s + 5\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied. This can be more clearly seen in Table 17.

Table 17: General solution for v = 18s + 3

$S_1 : r = 0$	1, 2, 8s + 3
r = 1	0, 4, 8s + 4
...	...
r = s − 1	−s + 2, 2s, 9s + 2
S2 : r = 0	5s + 1, 5s + 3, 9s + 3
r = 1	5s, 5s + 5, 9s + 4
...	...
r = s − 2	4s + 3, 7s − 1, 10s + 1
r = s − 1	4s + 2, 7s + 1, 10s + 2
S3 : r = 0	7s, 7s + 3, 13s + 2
...	...
r = s − 3	5s + 4, 8s + 1, 12s + 4
r = s − 2	5s + 2, 8s + 2, 12s + 3
6s + 1	8s + 3, 14s + 4, 20s + 5

Then, observe that:

$$\Delta(S_1) = \{\{3r + 1, 8s - r + 1, 8s + 2r + 2\} \mid 0 \le r \le s - 1\};$$
$$\Delta(S_2) = \{\{3r + 2, 4s - r, 4s + 2r + 2\} \mid 0 \le r \le s - 1\}; \quad (17)$$
$$\Delta(S_3) = \{\{3r + 3, 6s - 2r - 1, 6s + r + 2\} \mid 0 \le r \le s - 1\};$$
$$\Delta(S_4) = \{\{6s + 1\}\}.$$

From Lemma 3.6, condition (1) is also satisfied.

Lemma 5.2. Let v ≡ 9 (mod 18) and v ≠ 9. Then there exists a set S of (v − 3)/6 triples such that:

(1) Δ(S) is the set {1, 2, . . . , (v − 1)/2};

(2) The set S is near-regular.

Proof. Let v = 18s + 9.

Do the cases s = 1, 2 and 3.

Suppose first that s = 1. Let

$$S = \{\{1, 13, 14\}, \{0, 2, 7\}, \{5, 8, 16\}, \{11, 15, 21\}, \{10, 19, 28\}\}. \quad (18)$$

Then

$$\Delta(S) = \{1, 12, 13\} \cup \{2, 5, 7\} \cup \{3, 8, 11\} \cup \{4, 6, 10\} \cup \{9\}. \quad (19)$$

Suppose that s = 2. Let

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$$S=\{\{1,12,13\},\{0,2,19\},\{3,6,26\},\{4,8,18\},\{10,15,23\},\{11,17,35\},\{9,16,25\},\{7,22,37\}\}. \quad (20)$$

Then

$$\Delta(S)=\{1,11,12\} \cup \{2,17,19\} \cup \{3,20,22\} \cup \{4,10,14\} \cup \{5,8,13\} \cup \{6,18,21\} \cup \{7,9,16\} \cup \{15\}. \quad (21)$$

Suppose that $s = 3$. Let

$$S = \{\{1, 16, 17\}, \{0, 2, 29\}, \{3, 6, 31\}, \{4, 8, 22\}, \{5, 10, 36\}, \{7, 13, 30\}, \{11, 18, 31\}, \{14, 23, 34\}, \{12, 21, 45\}, \{15, 25, 37\}, \{19, 40, 61\}\}. \quad (22)$$

Then

$$\Delta(S) = \{1, 15, 16\} \cup \{2, 27, 29\} \cup \{3, 25, 28\} \cup \{4, 14, 18\} \cup \{5, 26, 31\} \cup \{6, 17, 23\} \cup \{7,13,20\} \cup \{8, 11, 19\} \cup \{9, 24, 30\} \cup \{10, 12, 22\} \cup \{21\}. \quad (23)$$

Otherwise, $s \geq 4$.

$$S_1 = \{\{1 - r, 2 + 2r, 4s + 5 + r\} \mid 0 \leq r \leq s\};$$
$$S_2 = \{\{5s + 12 - 2r, 8s + 2 + 2r, 16s + 2 + 2r\} \mid 0 \leq r \leq s - 2\}; \quad (24)$$
$$S_3 = \{\{10s + 6 - 2r, 10s + 10 - r, 16s + 11 - r\} \mid 0 \leq r \leq s - 2\};$$
$$S_4 = \{\{3s + 4, 5s + 6, 14s + 6\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied.

This can be more clearly seen in Table 18.

Table 18: General solution of v = 18s + 9

$S_1 : r = 0$	$1, 2, 4s+5$
$r = 1$	$0, 4, 4s+6$
...	...
$r = s$	$-s+1, 2s+2, 5s+5$
$S_2 : r = 0$	$8s+9, 8s+11, 16s+14$
$r = 1$	$8s+8, 8s+13, 16s+15$
$r = 2$	$5s+12, 8s+2, 16s+2$
...	...
$r = s-3$	$5s+10, 8s+6, 16s+4$
$r = s-2$	$8s+7, 11s+6, 14s+8$
$S_3 : r = 0$	$10s+8, 10s+11, 18s+12$
$r = 1$	$10s+6, 10s+10, 16s+11$
...	...
$r = s-3$	$10s+2, 11s+8, 16s+8$
$r = s-2$	$10s, 11s+9, 16s+7$
$3s, 7s+3, 8s+6$	$7, 3s+7, 8s+13$
$6s+3$	$4s, 10s+3, 16s+6$

Then, observe that:

$$\Delta(S_1) = \{\{3r + 1, 4s - r + 3, 4s + 2r + 4\} \mid 0 \leq r \leq s\};$$
$$\Delta(S_2) = \{\{3r + 2, 8s - r + 2, 8s + 2r + 4\} \mid 0 \leq r \leq s - 2\}; \quad (25)$$
$$\Delta(S_3) = \{\{3r + 3, 6s - 2r + 1, 6s + r + 4\} \mid 0 \leq r \leq s - 2\};$$
$$\Delta(S_4) = \{\{3s, 7s + 3, 8s + 6\}\}.$$

From Lemma 3.8, Condition (1) is also satisfied.

Lemma 5.3. Let $v \equiv 15 \pmod{18}$. Then there exists a set S of $(v - 3)/6$ triples such that:

(1) $\Delta(S)$ is the set $\{1, 2, . . . , (v - 1)/2\}$;

(2) The set S is near-regular.

Proof. Let $v = 18s + 15$.

First consider the case $s = 1$. Let $S = \{\{1, 2, 9\}, \{0, 4, 10\}, \{3, 5, 19\}, \{6, 11, 24\}, \{9, 12, 21\}, \{7, 18, 29\}\}$. Then

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$$\Delta(S) = \{1, 7, 8\} \cup \{4, 6, 10\} \cup \{2, 14, 16\} \cup \{5, 13, 15\} \cup \{3, 9, 12\} \cup \{11\}. \quad (26)$$

Otherwise, $s \geq 2$. Let

$$S_1 = \{\{1 - r, 2 + 2r, 4s + 5 + r\} \mid 0 \leq r \leq s\};$$
$$S_2 = \{\{6s + 6 - r, 6s + 8 + 2r, 14s + 14 + r\} \mid 0 \leq r \leq s\}; \quad (27)$$
$$S_3 = \{\{10s + 8 - 2r, 10s + 11 + r, 16s + 14 - r\} \mid 0 \leq r \leq s - 1\};$$
$$S_4 = \{\{6s + 7, 12s + 12, 18s + 13\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied.

This can be more clearly seen in Table 19.

Table 19: General solution of $v = 18s + 15$

$S1 : r = 0$	$1, 2, 4s + 5$
$r = 1$	$0, 4, 4s + 6$
...	...
$r = s$	$-s + 1, 2s + 2, 5s + 5$
$S2 : r = 0$	$6s + 6, 6s + 8, 14s + 14$
$r = 1$	$6s + 5, 6s + 10, 14s + 15$
...	...
$r = s - 1$	$5s + 7, 8s + 6, 15s + 13$
$r = s$	$5s + 6, 8s + 8, 15s + 14$
$S3 : r = 0$	$10s + 8, 10s + 11, 16s + 14$
...	...
$r = s - 2$	$8s + 12, 11s + 9, 15s + 16$
$r = s - 1$	$8s + 10, 11s + 10, 15s + 15$
$6s + 5$	$6s + 7, 12s + 12, 18s + 13$

Then, observe that:

$$\Delta(S_1) = \{\{3r + 1, 4s - r + 3, 4s + 2r + 4\} \mid 0 \leq r \leq s\};$$
$$\Delta(S_2) = \{\{3r + 2, 8s - r + 6, 8s + 2r + 8\} \mid 0 \leq r \leq s\}; \quad (28)$$
$$\Delta(S_3) = \{\{3r + 3, 6s - 2r + 3, 6s + r + 6\} \mid 0 \leq r \leq s - 1\};$$
$$\Delta(S_4) = \{\{6s + 5\}\}.$$

From Lemma 3.10, Condition (1) is also satisfied.

In this way, section 4 and section 5 proved all the Lemmas obtained by all types of Heffter's Difference Problem. At the same time the parameters in the tables become one of the expressions of the solutions that can be applied directly.

6. Conclusion

The Heffter difference problem is a classic problem in combinatorics, its solutions are often used to construct various combinatorial designs (such as block designs, Latin squares, etc.) and are cited here as a construction tool. In this paper, the row-column design constructed here forms maintains the core characteristics based on the Balanced Incomplete Block Design, and clarifies the regularity of rows and the near-regularity of columns. Those constructions make use of known solutions to Heffter's Difference Problem.

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