ISSN 2616-5805 Vol. 7, Issue 1: 56-62, DOI: 10.25236/AJMS.2026.070108

A Discrete Fractional Duopoly Game with Bounded Rationality and Consumer Surplus

Jianglin Zhao*

School of Mathematics, Sichuan Minzu College, Kangding, China
ws05101162@163.com
**Corresponding author*

Abstract: *A discrete fractional-order Cournot duopoly game with consumer surplus (CS) is proposed to capture long-memory effects in firms' output decisions, reflecting the realistic reliance on historical information. The objective of each firm is a weighted average of its own profit and consumer surplus, where the weight indicates the degree of public ownership. Four Nash equilibria are identified: a trivial equilibrium, two corner monopoly equilibria, and a unique interior duopoly equilibrium. Local stability analysis via linearization shows that the trivial and corner equilibria are always unstable, while explicit stability conditions for the interior equilibrium are derived in terms of the memory intensity, the public ownership weight, and the adjustment speed. Numerical simulations, including stability diagrams and phase portraits, validate the theoretical results and demonstrate that a higher degree of public ownership narrows the stable range of adjustment speeds. The introduction of fractional-order dynamics not only enriches the model's realism but also imposes stricter stability thresholds compared to existing integer-order CS models, thereby bridging the gap between long-memory modeling and social-welfare-oriented duopoly games.*

Keywords: *Cournot duopoly; consumer surplus; stability; long-memory effect*

1. Introduction

Oligopoly, a market structure between perfect competition and monopoly, is characterized by strategic interaction among few large firms. Since Cournot [1] introduced quantity competition, dynamic extensions have incorporated bounded rationality [2] and nonlinear dynamics [3] to better mimic real-world decision-making.

In practice, many firms—especially state-owned or mixed enterprises—pursue not only profit but also social welfare. Consumer surplus (CS) is a key welfare measure. Recent works have integrated CS into duopoly games: Cao et al. [4] studied global dynamics and synchronization in a bounded-rational Cournot model with CS, while Zhou et al. [5] performed detailed bifurcation analysis (flip and Neimark–Sacker) using center manifold and normal form theory. However, both models are built on integer-order difference equations, which ignore long-memory effects in firms' output decisions.

Real firms often rely on historical data over multiple past periods—a long-memory feature. Integer-order difference equations are Markovian and cannot capture this. Discrete fractional calculus, with the fractional order governing memory intensity, naturally accommodates such effects. Xin et al. [6] pioneered the use of discrete fractional-order difference equations in a Cournot duopoly, demonstrating bifurcations and chaos via the 0-1 test. Nevertheless, their model does not include CS or mixed ownership.

To date, no study combines discrete fractional-order dynamics with CS-augmented objectives in a duopoly game. Existing CS models [4, 5] lack memory effects, while fractional-order models [6] omit consumer surplus. This paper fills that gap by integrating both features.

The remainder is organized as follows: Section 2 recalls preliminaries of fractional calculus; Section 3 formulates the model; Section 4 analyzes stability; Section 5 presents numerical simulations; Section 6 concludes.

ISSN 2616-5805 Vol. 7, Issue 1: 56-62, DOI: 10.25236/AJMS.2026.070108

2. Preliminaries

Definition 2.1. ([7]) For any real numbers υ, the υ-th falling fractional factorial of a real number $t \in \mathbb{R}$ is defined as

$$t^{(\upsilon)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\upsilon)},\ t^{(0)} = 1. \tag{1}$$

Definition 2.2. ([7]) Let $f: \mathbb{N}_a \to \mathbb{R}$ and $\upsilon > 0$. The υ-th order fractional sum is defined as

$$\Delta_a^{-\upsilon} f(t) = \frac{1}{\Gamma(\upsilon)} \sum_{s=a}^{t-\upsilon} (t-s-1)^{(\upsilon-1)} f(s), \tag{2}$$

where a is the starting point, $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$ denotes the discrete time scale, and $t \in \mathbb{N}_{a+\upsilon}$.

Definition 2.3. ([8]) Let $f: \mathbb{N}_a \to \mathbb{R}$, $\upsilon > 0$ and $m = \lceil \upsilon \rceil$. The Caputo difference is defined as

$${}^{C}\Delta_a^{\upsilon} f(t) = \begin{cases} \Delta_a^{-(m-\upsilon)} \Delta^m f(t), & m-1 < \upsilon < m, \\ \Delta^m f(t), & \upsilon = m, \end{cases} \tag{3}$$

where $t \in \mathbb{N}_{a+m-\upsilon}$.

Theorem 2.1. ([9]) Let $0 < \upsilon < 1$ and $A \in \mathbb{R}^{d \times d}$. The Caputo fractional difference equation

$${}^{C}\Delta_a^{\upsilon} x(t) = Ax(t+\upsilon-1),\ t \in \mathbb{N}_{a+1-\upsilon},$$

is asymptotically stable if all eigenvalues $\lambda_i (i = 1, 2, \cdots, d)$ of A satisfy

$$\lambda_i \in \left\{ z \in \mathbb{C} \middle| |z| < \left(2\cos\frac{|\arg z| - \pi}{2-\upsilon} \right)^{\upsilon}, |\arg z| > \frac{\upsilon\pi}{2} \right\}.$$

In particular, if all eigenvalues λ_i of A are real, then the stability condition reduces to

$$\lambda_i \in (-2^{\upsilon}, 0).$$

3. Model Formulation

A classical Cournot duopoly with two firms producing homogeneous and perfectly substitutable products is considered. Let $q_i(t)$ denote the output of firm $i (i = 1, 2)$ at discrete period $t \in \mathbb{N}_0$. The market-clearing price is assumed to be determined by a linear inverse demand function given by

$$p(t) = \alpha - \beta(q_1(t) + q_2(t)), \tag{4}$$

where $\alpha > 0$ denotes the maximum market price and $\beta > 0$ represents the price decrease per unit increase in total output.

Both firms are assumed to have constant and identical marginal production costs. The cost function of firm i can thus be expressed as

$$C_i(t) = cq_i(t),\ i = 1, 2, \tag{5}$$

where $0 < c < \alpha$ denotes the identical marginal cost.

Accordingly, the instantaneous profit for firm i is calculated as

$$\Pi_i(t) = \big(p(t) - c\big) q_i(t) = \big(\alpha - \beta(q_1(t) + q_2(t)) - c\big) q_i(t),\ i = 1, 2. \tag{6}$$

Consumer surplus is defined as the area below the demand curve and above the market price line. For the linear demand function, it is derived as

$$CS(t) = \int_{p(t)}^{\alpha} \frac{\alpha - \overline{p}}{\beta} d\overline{p} = \frac{\beta\big(q_1(t) + q_2(t)\big)^2}{2}, \tag{7}$$

where $\overline{p} \in (p(t), \alpha)$ denotes the price variable.

ISSN 2616-5805 Vol. 7, Issue 1: 56-62, DOI: 10.25236/AJMS.2026.070108

In a mixed oligopoly, the objectives of firms often combine private profit with social welfare. Both firms are assumed to possess an identical degree of public ownership, denoted by $\mu \in (0,1)$, which serves as the weight assigned to consumer surplus. Consequently, the objective function for firm i is given by the weighted sum of its profit and consumer surplus:

$$O_i\left(q_1(t), q_2(t)\right) = (1-\mu)\Pi_i(t) + \mu CS(t)\ , i = 1,2\ . \tag{8}$$

Specifically, $\mu = 0$ indicates a purely private enterprise focused on maximizing profit, while $\mu = 1$ represents a fully public enterprise maximizing social welfare. For $0 < \mu < 1$, the firm features mixed ownership and balances private profitability with social responsibility.

The marginal objective for firm i is obtained by taking the partial derivative of O_i with respect to its own output:

$$\phi_i(t) = \left.\frac{\partial O_i(q_1, q_2)}{\partial q_i}\right|_{(q_1(t), q_2(t))} = (1-\mu)(\alpha - c) - \beta\left[(2-3\mu)q_i(t) + (1-2\mu)q_j(t)\right],\ i, j = 1,2\ ,\ i \neq j\ . \tag{9}$$

In the bounded rational repeated game considered here, firms integrate historical output information into their current production decisions. Both firms are assumed to employ identical decision-making rules and share the same adjustment speed. Utilizing the Caputo fractional difference operator, the fractional-order dynamical adjustment mechanism with long-memory effect is described by

$$\begin{cases} {}^{C}\Delta_0^{\upsilon} q_1(t) = \delta q_1(t+\upsilon-1)\phi_1(t+\upsilon-1), \\ {}^{C}\Delta_0^{\upsilon} q_2(t) = \delta q_2(t+\upsilon-1)\phi_2(t+\upsilon-1), \end{cases} \tag{10}$$

where $\delta > 0$ is the adjustment speed of the two firms, $\upsilon \in (0,1)$ is the fractional order characterizing the memory intensity, $t \in \mathbb{N}_{1-\upsilon}$, and $a = 0$. As the fractional operator introduces a non-local effect, the output $q_i(t)$ depends not only on the current marginal objective but also on the historical states of the system.

The influence of past states on the current state is explicitly captured by the following explicit scheme, which allows system (10) to be rewritten as

$$\begin{cases} q_1(n) = q_1(0) + \dfrac{\delta}{\Gamma(\upsilon)} \displaystyle\sum_{i=1}^{n} \frac{\Gamma(n-i+\upsilon)}{\Gamma(n-i+1)} q_1(i-1)\left[(1-\mu)(\alpha-c) + (3\mu-2)\beta q_1(i-1) + (2\mu-1)\beta q_2(i-1)\right], \\ q_2(n) = q_2(0) + \dfrac{\delta}{\Gamma(\upsilon)} \displaystyle\sum_{i=1}^{n} \frac{\Gamma(n-i+\upsilon)}{\Gamma(n-i+1)} q_2(i-1)\left[(1-\mu)(\alpha-c) + (3\mu-2)\beta q_2(i-1) + (2\mu-1)\beta q_1(i-1)\right]. \end{cases} \tag{11}$$

4. Local Stability

The equilibrium points of the model (10) satisfy

$$\begin{cases} q_1\left[(1-\mu)(\alpha-c) + (3\mu-2)\beta q_1 + (2\mu-1)\beta q_2\right] = 0, \\ q_2\left[(1-\mu)(\alpha-c) + (3\mu-2)\beta q_2 + (2\mu-1)\beta q_1\right] = 0. \end{cases} \tag{12}$$

There are four Nash equilibrium points as follows:

(i) $E_0 = (0,0)$ always exists. At the trivial equilibrium E_0, if one firm produces nothing, the other cannot profit from any positive output. Thus, zero production is the unique best response for both.

(ii) When $0 < \mu < \frac{2}{3}$, $E_1 = \left(0, \frac{(1-\mu)(\alpha-c)}{\beta(2-3\mu)}\right)$ exists. At the monopoly equilibrium E_1, Firm 1's optimal response to firm 2's equilibrium output is to produce nothing; conversely, firm 2's equilibrium output is optimal when firm 1 produces zero. Hence, this is a corner equilibrium where firm 2 acts as a monopolist.

(iii) When $0 < \mu < \frac{2}{3}$, $E_2 = \left(\frac{(1-\mu)(\alpha-c)}{\beta(2-3\mu)}, 0\right)$ exists. At the monopoly equilibrium E_2, symmetrically, firm 2's best response to firm 1's equilibrium output is zero output, and firm 1's equilibrium output is optimal when firm 2 produces nothing. Firm 1 becomes the sole producer.

ISSN 2616-5805 Vol. 7, Issue 1: 56-62, DOI: 10.25236/AJMS.2026.070108

(iv) When $0<\mu<\frac{3}{5}$, $E^*=\left(\frac{(1-\mu)(\alpha-c)}{\beta(3-5\mu)},\frac{(1-\mu)(\alpha-c)}{\beta(3-5\mu)}\right)$ exists. At the duopoly equilibrium E^*, at this interior equilibrium, neither firm can increase its payoff by unilaterally deviating from its prescribed output, given that the other firm maintains its equilibrium strategy. Both firms produce positive quantities, reflecting a simultaneous best-response outcome.

Theorem 4.1. E_0 is unstable.

Proof. The Jacobian matrix $J(E_0)$ of system (10) at E_0 is

$$J(E_0)=\begin{bmatrix}\delta(1-\mu)(\alpha-c) & 0\\ 0 & \delta(1-\mu)(\alpha-c)\end{bmatrix}.$$

The eigenvalues of $J(E_0)$ are $\lambda_i=\delta(1-\mu)(\alpha-c)>0, i=1,2$. According to Theorem 1, the equilibrium point E_0 is unstable.

Theorem 4.2. E_1 is unstable.

Proof. The Jacobian matrix $J(E_1)$ of system (10) at E_1 is

$$J(E_1)=\begin{bmatrix}\frac{\delta(1-\mu)^2(\alpha-c)}{2-3\mu} & 0\\ \frac{\delta(2\mu-1)(1-\mu)(\alpha-c)}{2-3\mu} & -\delta(1-\mu)(\alpha-c)\end{bmatrix},$$

whose eigenvalues are

$$\lambda_1=\frac{\delta(1-\mu)^2(\alpha-c)}{2-3\mu}>0,\ \lambda_2=-\delta(1-\mu)(\alpha-c)<0.$$

According to Theorem 1, the equilibrium point E_1 is unstable.

Theorem 4.3. E_2 is unstable.

Proof. The Jacobian matrix $J(E_2)$ of system (10) at E_2 is

$$J(E_2)=\begin{bmatrix}-\delta(1-\mu)(\alpha-c) & \frac{\delta(2\mu-1)(1-\mu)(\alpha-c)}{2-3\mu}\\ 0 & \frac{\delta(1-\mu)^2(\alpha-c)}{2-3\mu}\end{bmatrix},$$

whose eigenvalues are

$$\lambda_1=-\delta(1-\mu)(\alpha-c)<0,\ \lambda_2=\frac{\delta(1-\mu)^2(\alpha-c)}{2-3\mu}>0.$$

According to Theorem 1, the equilibrium point E2 is unstable.

Theorem 4.4. E^* is locally asymptotically stable if one of the following conditions holds:

(i) When $0<\mu\le\frac{1}{2}$, $0<\delta<\frac{2^{\upsilon}}{(1-\mu)(\alpha-c)}$;

(ii) When $\frac{1}{2}<\mu<\frac{3}{5}$, $0<\delta<\frac{2^{\upsilon}(3-5\mu)}{(1-\mu)^2(\alpha-c)}$.

Proof. The Jacobian matrix $J(E^*)$ of system (10) at E^* is

$$J(E^*)=\frac{\delta(1-\mu)(\alpha-c)}{(3-5\mu)}\begin{bmatrix}3\mu-2 & 2\mu-1\\ 2\mu-1 & 3\mu-2\end{bmatrix},$$

whose eigenvalues are

ISSN 2616-5805 Vol. 7, Issue 1: 56-62, DOI: 10.25236/AJMS.2026.070108

$$\lambda_1 = -\delta(1-\mu)(\alpha-c) < 0,\ \lambda_2 = -\frac{\delta(1-\mu)^2(\alpha-c)}{3-5\mu} < 0.$$

From Theorem 1, the local stability condition for E^* is obtained as

$$\max\left\{\delta(1-\mu)(\alpha-c), \frac{\delta(1-\mu)^2(\alpha-c)}{3-5\mu}\right\} < 2^{\upsilon}.$$

Then it can be deduced that

$$0 < \delta < \min\left\{\frac{2^{\upsilon}}{(1-\mu)(\alpha-c)}, \frac{2^{\upsilon}(3-5\mu)}{(1-\mu)^2(\alpha-c)}\right\}.$$

Furthermore, when $0 < \mu \le \frac{1}{2}$, $0 < \delta < \frac{2^{\upsilon}}{(1-\mu)(\alpha-c)}$; when $\frac{1}{2} < \mu \le \frac{3}{5}$, $0 < \delta < \frac{2^{\upsilon}(3-5\mu)}{(1-\mu)^2(\alpha-c)}$.

5. Numerical Simulation

To corroborate the analytical stability conditions established in Theorem 4.4, the explicit fractional-order difference scheme (11) is employed for all numerical experiments. The parameters are set as $\alpha - c = 1$ and $\beta = 1$ without loss of generality, while the fractional order υ, the public ownership weight μ, and the adjustment speed δ are varied according to the theoretical bounds.

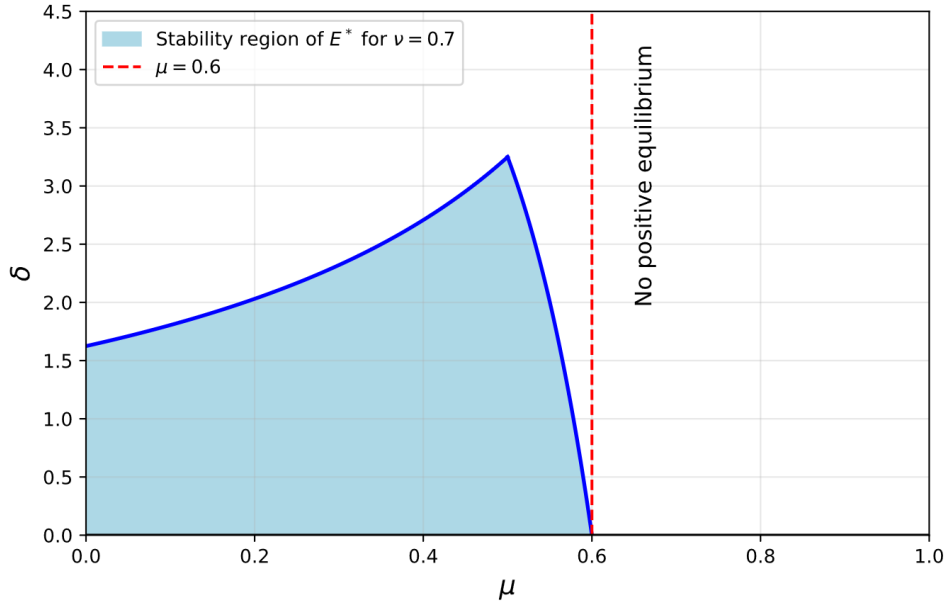


Figure 1: Stability region of E^* in the (μ,δ) -parameter plane.

Figure 1 depicts the stability region of the interior Nash equilibrium E^* in the (μ,δ) parameter plane for a fixed memory intensity $\upsilon = 0.7$. The solid boundary curves are determined by the critical values derived in Theorem 4.4: for $0 < \mu \le \frac{1}{2}$, the upper stability limit is $\delta = \frac{2^{\upsilon}}{(1-\mu)(\alpha-c)}$; for $\frac{1}{2} < \mu < \frac{3}{5}$, the limit becomes $\delta = \frac{2^{\upsilon}(3-5\mu)}{(1-\mu)^2(\alpha-c)}$. The shaded area below these curves indicates the parameter combinations for which E^* is locally asymptotically stable. An evident monotonic relation is observed: as μ increases, the permissible range of δ narrows, implying that a stronger public ownership orientation reduces the capacity for rapid output adjustments without destabilizing the equilibrium.

A representative parameter set inside the stable region is selected as $\upsilon = 0.7$, $\mu = 0.3$, and $\delta = 0.5$, which satisfies the stability inequality since $\delta = 0.5 < 2^{0.7} / 0.7 \approx 2.32$. Figure 2 presents the phase portrait of system (10) under this parameter configuration, with several trajectories initiated from distinct initial

ISSN 2616-5805 Vol. 7, Issue 1: 56-62, DOI: 10.25236/AJMS.2026.070108

conditions. All plotted orbits converge asymptotically to the interior fixed point E^*, highlighted by a red star. This convergence behavior is in full accordance with the theoretical prediction and visually confirms the local asymptotic stability of the Nash equilibrium.

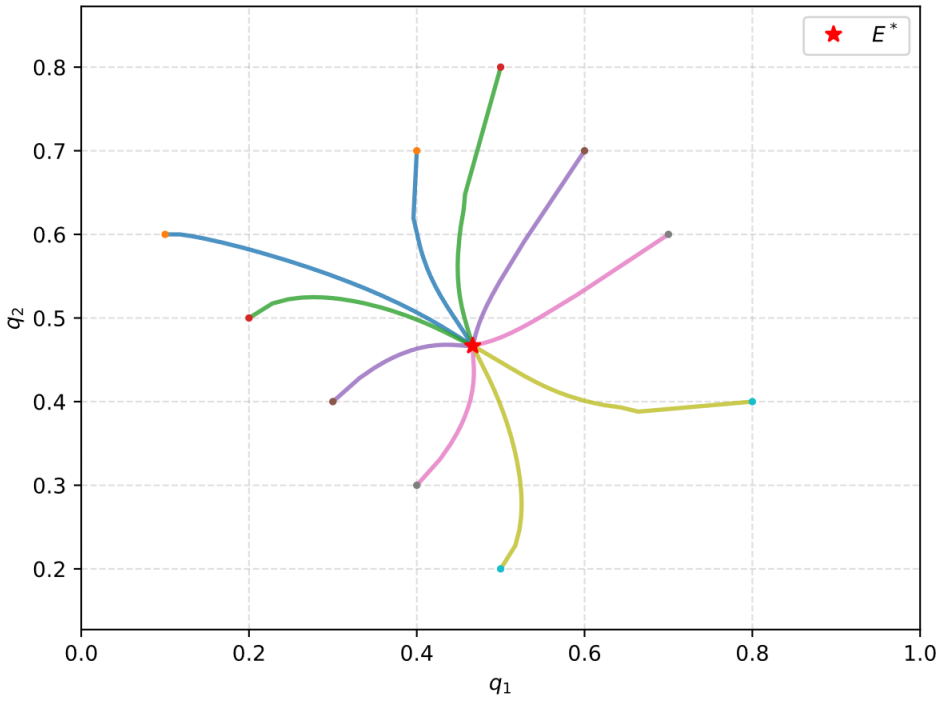


Figure 2: Phase portrait of system (10). Trajectories from different initial conditions tend to E^* (red star).

Together, these numerical results not only validate the derived stability criteria but also illustrate the coupled effects of memory intensity, public ownership weight, and adjustment speed on the long-term dynamics of the fractional-order Cournot duopoly game with consumer surplus.

6. Conclusion

This paper has proposed a discrete fractional-order Cournot duopoly game that incorporates consumer surplus into the objective functions of boundedly rational firms. By employing the Caputo-type fractional difference operator, the model captures long-memory effects in output decisions, a feature absent in classical integer-order formulations. The main analytical results establish the existence of four Nash equilibria, among which the trivial and monopoly corner equilibria are shown to be unstable for all admissible parameter values. For the interior duopoly equilibrium E^*, explicit local stability conditions are derived in terms of the fractional order υ, the public ownership weight μ, and the adjustment speed δ. Specifically, the stability region is characterized by upper bounds on E^* that depend on μ and υ, revealing that a higher degree of public ownership narrows the range of stable adjustment speeds.

Numerical simulations, including stability diagrams and phase portraits, fully corroborate the theoretical findings. In particular, Figure 1 delineates the stable parameter region in the (μ,δ) plane for $\upsilon=0.7$, and Figure 2 demonstrates the asymptotic convergence of trajectories to E^* under a stable parameter setting. These results confirm that the inclusion of fractional-order dynamics not only enriches the model's realism but also alters the stability thresholds compared to integer-order CS models ([4, 5]). Unlike the purely fractional-order model of Xin et al. [6], which ignores consumer surplus, the present framework simultaneously addresses both long-memory and social-welfare objectives.

The findings carry practical implications: policymakers and firm managers should be aware that stronger public ownership may require more conservative output adjustment strategies to maintain market stability. Future research may extend this work to heterogeneous adjustment speeds,

ISSN 2616-5805 Vol. 7, Issue 1: 56-62, DOI: 10.25236/AJMS.2026.070108

asymmetric ownership weights, or stochastic fractional-order environments, and may also explore bifurcation and chaos phenomena using the theoretical tools developed here.

Acknowledgement

This work was supported by the Research Center for Mathematics and Finance, a Key Research Base of Dazhou Federation of Social Sciences (Project Name: Research on Fractional-Order Duopoly Models Incorporating Corporate Social Responsibility, Project No. SCMF202407).

References

[1] Cournot, A. Researches into the mathematical principles of the theory of wealth, New York: The Macmillan Company, 1897.
[2] Bischi, G. I., Naimzada, A. Global analysis of a dynamic duopoly game with bounded rationality, Adv. Dyn. Games Appl. 2000, 5, 361–385.
[3] Rand, D. Exotic phenomena in games and duopoly models, J. Math. Econ. 1978, 5, 173–184.
[4] Cao, Y., Zhou, W., Chu, T. Global dynamics and synchronization in a duopoly game with bounded rationality and consumer surplus. International Journal of Bifurcation and Chaos, 2019, 29(11), 1930031.
[5] Zhou, W., Cao, Y., Elsonbaty, A., Elsadany, A. A., Chu, T. Bifurcation analysis of a bounded rational duopoly game with consumer surplus. International Journal of Bifurcation and Chaos, 2021, 31(7), 2150097.
[6] Xin, B., Peng, W., Kwon, Y. A discrete fractional-order Cournot duopoly game. Physica A: Statistical Mechanics and its Applications, 2020, 558, 124993.
[7] Atici F. M., Eloe P. W. Initial value problems in discrete fractional calculus. Proc Amer Math Soc 2009, 137(3):981–9.
[8] Abdeljawad T. On Riemann and caputo fractional differences, Comput. Math. Appl. 2011,62: 1602–1611.
[9] Čermák J, Györi I, Nechvátal L. On explicit stability conditions for a linear fractional difference system. Fract Calc Appl Anal 2015, 18(3):651–72.