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Discussion on the correspondence between nonlinear expectations and financial risk measurements

Xiaobo Wen*

Sichuan Minzu College, Kangding, Sichuan, 626001, China
541248034@qq.com

Abstract: *Under the axiomatic assumption of nonlinear expectations and the basic framework of financial risk measurement, the relationship between expectations such as sub-linear expectation, sub-additive expectation, convex (concave) expectation, and financial risk measurements such as risk measurement, consistent risk measurement, and convex risk measurement is established. Furthermore, the equivalent replacements of some properties such as sub-additive, super-additive, convexity, and concaveness in the relevant definitions and theorems are deduced, and the conditions in some definitions and theorems are relaxed.*

Keywords: *nonlinear expectation; sub-linear; convex; coherent risk measure; convex risk measurement*

1. Introduction

In financial mathematics, risk theory often does not satisfy simple linear relationships. It is often difficult to obtain the desired results by studying the risk theory in financial mathematics based on the traditional linear mathematical expectations. Since Nobel laureate in economics ALLAIS proposed the famous ALLAIS paradox, the traditional theory of linear expectation utility has been greatly challenged. Relevant studies have shown that the linear nature of linear mathematical expectations is the main reason for the emergence of ALLAIS paradox, so scholars have derived the theory of nonlinear mathematical expectations without linearity attached to the theory of mathematical expectations, and have studied many theoretical results of nonlinear mathematical expectations.

In 1997, PENG [1] introduced a typical class of domain-flow-compatible nonlinear mathematical expectations-g-expectations using inverted stochastic differential equations. COQUET [2] proposes an axiomatic definition of nonlinear mathematical expectations compatible with information flow and then studies the relationship between such nonlinear mathematical expectations and g-expectations. PENG [3] proposed the axiomatic concept of nonlinear mathematical expectations. In 2009, JIA [4] derived some properties of the extremely small elements in sub-linear expectations through set theory. PENG [5] reviews the detailed framework structure in nonlinear mathematical expectations and systematically expounds on the theory of nonlinear expectations. In the financial field, financial risk has also been described more through theories such as risk measurement, consistent risk measurement, convex risk measurement, consistent convex risk measurement, and dynamic risk measurement. ARTENER [6] proposed the concept of consistent risk measurement. In 2002, FOLLMER [7] and FRITTELLI [8] independently proposed the concept of convex risk measurement. DETLEFSEK K [9] introduces the definition of the conditional convexity risk measure, obtaining sufficient necessary conditions that the conditional convexity risk measure can represent. Robert and Rogers [10] extend static risk metrics to dynamic risk metrics. Literature [11-12] examines static portfolios and risk measurement.

This paper aims to study the relationship between various mathematical expectations and between risk measurement and mathematical expectations. There are also some scholars in this field who have done some research. ROSAZZA GIANINE [13] ⁱnvestigates the intrinsic link between the g-expectation theory and its induced risk measurement. JEANG L [14] Studies the intrinsic relationship between nonlinear mathematical expectations and financial risk measurements within the basic framework of axiomatic assumptions. OELBAEN F, PENG SG, ROSAZZA G E [15] Representation of dynamic convexity risk measures induced by g-expectations gives the representation results of a class of time-compatible dynamic convex risk measures (dynamic, concave utility). In 2020, Huang [16] explored the relationship between sub-linear expectations and consistent risk measures, convex expectations, and convex risk measures.

In this paper, some property transformations and equality issues in the expectation field are further

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explored. In the sub-linear space, negative homogeneous and sub-linear conditions can be derived together to derive linear relationships. In a sub-linear space, based on satisfying the homogeneous, the sub-(super) additive can satisfy the convexity (concave). The conjugate that yields a sub-linear (super)linear expectation is a super(sub)linear expectation. This paper obtains some of the intrinsic connections between nonlinear mathematical expectations and financial risk measurements.

In the relationship of the $\rho(X)=-\varepsilon[X]$ function, when the ε is a linear expectation, ρ is a risk measure. In the relationship of the $\rho(X)=\varepsilon[-X]$ function, if the ε is a risk measure, then ρ is a nonlinear mathematical expectation. Under the relationship of the $\rho(X)=\varepsilon[-X]$ function, if ρ is a consistent risk measure, then the ε is secondary plus expectation. Under the relationship of the $\rho(X)=\varepsilon[-X]$ function, if ρ is a consistent risk measure, then the ε is secondary plus expectation. Under the $\rho(X)=\varepsilon[-X]$ function relationship, when the ε is convex, add any of the following conditions (i.), (ii.), (iii.) to prove that ρ is a convex risk measure. (i.) sub-additive: $\varepsilon(X+\mathrm{c})\le\varepsilon(X)+\mathrm{c}$, (ii.) positive homogeneity: $\varepsilon(\beta X)=\beta\varepsilon(X),\beta>0$,(iii.) if $\varepsilon[(1-\lambda)X+\mathrm{c}]\to\varepsilon(X+\mathrm{c})$, where $\lambda\to 0$ but $\lambda\neq 0$. In the relationship of the $\rho(X)=-\varepsilon[X]$ function, when the ε is concave, any of the following conditions (1), (2), (3) can be proved ρ to be a convex risk measure. (1) Super-additive: $\varepsilon(X+\mathrm{c})\ge\varepsilon(X)+\mathrm{c}$, (2) positive homogeneity: $\varepsilon(\beta X)=\beta\varepsilon(X),\beta>0$, (3) if $\varepsilon[(1-\lambda)X+\mathrm{c}]\to\varepsilon(X+\mathrm{c})$, where $\lambda\to 0$ but $\lambda\neq 0$.

2. Prerequisite

Let (Λ,F,P) be the complete probability space, and let $L(\Lambda,F,P)$ be the total of the integral random variables. According to the relevant theories in the reference [4], the following definitions of expectations are given.

For the sake of simplicity, first, give some properties of the mathematical expectation field.

A0 Linearity: $\varepsilon[\alpha X+\beta Y]=\alpha\varepsilon(X)+\beta\varepsilon(Y),\forall X,Y\in L(\Lambda,F,P),\alpha,\beta\in R$.

A1 Permutation: $\varepsilon[c]=c,\forall c\in R.$

A2 Monotony: $\varepsilon[X]\ge\varepsilon[Y],\forall X\ge Y.$

A3 Sub-additive: $\varepsilon[X+Y]\le\varepsilon[X]+\varepsilon[Y].$

A4 Positive homogeneity: $\varepsilon[\lambda X]=\lambda\varepsilon[X],\forall\lambda\ge 0.$

A5.Super-additive: $\varepsilon[X+Y]\ge\varepsilon[X]+\varepsilon[Y].$

A6.Convexity: $\varepsilon[\lambda X+(1-\lambda)\mathrm{Y}]\le\lambda\varepsilon[X]+(1-\lambda)\varepsilon[Y],\forall\lambda\in[0,1]$

A7.Concave: $\varepsilon[\lambda X+(1-\lambda)\mathrm{Y}]\ge\lambda\varepsilon[X]+(1-\lambda)\varepsilon[Y],\forall\lambda\in[0,1]$

Definition 1 Define a Real-value functional $\varepsilon:L(\Lambda,F,P)\to R$ is a linear mathematical expectation if it satisfies A0, A1, A2.

Definition 2 Define a Real-value functional $\varepsilon:L(\Lambda,F,P)\to R$ is a nonlinear mathematical expectations if they satisfy A1, A2.

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Definition 3 Define a Real-value functional $\varepsilon : L(\Lambda, F, P) \to R$ is a sub-additive mathematical expectation if it satisfiesA1, A2, A3.

Definition 4 Define a Real-value functional $\varepsilon : L(\Lambda, F, P) \to R$ is a sub-linear mathematical expectations if they satisfy A1, A2, A3, A4.

Definition 5 Define a Real-value functional $\varepsilon : L(\Lambda, F, P) \to R$ is a super-additive mathematical expectation if it satisfies A1, A2, A5.

Definition 6 Define a Real-value functional $\varepsilon : L(\Lambda, F, P) \to R$ is a super-linear mathematical expectations if they satisfy A1, A2, A4, A5.

Definition 7 Define a Real-value functional $\varepsilon : L(\Lambda, F, P) \to R$ is a convex mathematical expectations if they satisfy A1, A2, A6.

Definition 8 Define a Real-value functional $\varepsilon : L(\Lambda, F, P) \to R$ is a concave mathematical expectation if it satisfies A1, A2, A7.

Classification of Common mathematical expectation as shown in Table 1.

Table 1 Classification of Common mathematical expectation

Mathematical expectation	Linear mathematical expectation		
	Nonlinear mathematical expectation	Convex mathematical expectation	
		Concave mathematical expectation	
		Sub-additive mathematical expectation	Sub-linear mathematical expectation
		Super-additive mathematical expectation	Super-linear mathematical expectation

Similarly, for the sake of simplicity, according to the relevant theories in the reference [6-8], some properties and definitions of the field of financial measurement are given.

B1 Inverse monotony: $\rho(X) \le \rho(Y), X \ge Y.$

B2 Translation invariant: $\rho(X + c) = \rho(X) - c, X \in L(\Lambda, F, P), c \in R.$

B3 Sub-additive: $\rho(X + Y) \le \rho(X) + \rho(Y), X, Y \in L(\Lambda, F, P).$

B4 Homogeneous: $\rho(\lambda X) = \lambda\rho(X), X \in L(\Lambda, F, P), \lambda \in [0,1]$.

B5 Convexity: $\rho[\lambda X + (1-\lambda)\mathrm{Y}] \le \lambda\rho[X] + (1-\lambda)\rho[X], X, Y \in L(\Lambda, F, P), \lambda \in [0,1]$.

B6 Normalization: $\rho(0) = 0$. (general initial conditions in the field of static risk measurement).

Definition 9 Define a functional $\rho : L(\Lambda, F, P) \to R$ is a risk measure if it satisfies B1, B2.

Definition 10 Define a functional $\rho : L(\Lambda, F, P) \to R$ is a consistent risk measure if it satisfies B1, B2, B3, B4.

Definition 11 Define a functional $\rho : L(\Lambda, F, P) \to R$ is a convex risk measure if it satisfies B1, B2, B5, B6.

Classification of Common risk measure as shown in Table 2.

Table 2 Common risk measure

Common risk measure	
Consistent risk measure	Convex risk measure

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3. Mathematical expectations of the relationship between the results and their proof

Theorem 1 In a sub-linear expected space, negative homogeneous + sub-linearity ⇔ linearity. Among them, the negative flush: $\varepsilon[-\lambda X]=-\lambda\varepsilon[X],\forall\lambda>0.$

Proof It proves that sub-linear spaces have permutation, Monotony, sub-additive, and homogeneity. By sub-additive $\varepsilon[\alpha X+\beta Y]\le\varepsilon(\alpha X)+\varepsilon(\beta Y)=\alpha\varepsilon(X)+\beta\varepsilon(Y),\alpha,\beta>0$, using sub-linear and negative homogeneous

$$\alpha\varepsilon(X)=\varepsilon(\alpha X)=\varepsilon[\alpha X+\beta Y-\beta Y]\le\varepsilon(\alpha X+\beta Y)+\varepsilon(-\beta Y)=\varepsilon(\alpha X+\beta Y)$$

$-\beta\varepsilon(Y)$ that is, $\alpha\varepsilon(X)+\beta\varepsilon(Y)\le\varepsilon[\alpha X+\beta Y]$,

in summary, $\varepsilon[\alpha X+\beta Y]=\alpha\varepsilon(X)+\beta\varepsilon(Y)$, linearity is demonstrated and vice versa.

Therefore, linear mathematical expectations can be regarded as sub-linear expectations with negative homogeneous conditions attached. If the space in which linear mathematical expectations are formed is E* and the space in which sub-linear mathematical expectations are formed is E#, then there is E* ⊂ E#, and JIA [4] also proves the validity of this conclusion from the perspective of set theory.

Theorem 2 In a nonlinear expectation space, if a nonlinear mathematical expectation satisfies the sub-additive and homogeneous sexes, then the expectation satisfies convexity.

Proof $\varepsilon[\lambda X+(1-\lambda)\mathrm{Y}]\le\varepsilon[\lambda X]+\varepsilon[(1-\lambda)Y]=\lambda\varepsilon[X]+(1-\lambda)\varepsilon[Y]\ ,\forall\lambda\in[0,1]$.

It can be seen from this that when the ε is a sub-linear expectation, the ε is also a convex expectation. However, if ε is expected to be secondary additive, since it is not homogeneous and the establishment of convexity cannot be deduced by satisfying the secondary additive. Since the sub-linear expectation has a homogeneous condition, the scope of use is limited, so the sub-additive expectation can also be regarded as a generalization of the sub-linear expectation.

Note 1: Interestingly, if it is known that convexity is true, on the basis that the homogeneity is also true, the sub-additive is also true at this time. At this time, the efficacy of convexity and the efficacy of secondary plus are consistent. It can be deduced that $\varepsilon[\lambda X+(1-\lambda)\mathrm{Y}]\le\lambda\varepsilon[X]+(1-\lambda)\varepsilon[Y]$, set $\forall\lambda=\frac{1}{2},1-\lambda=\frac{1}{2}$, so that $\varepsilon[\frac{1}{2}X+\frac{1}{2}\mathrm{Y}]=\frac{1}{2}\varepsilon[X+\mathrm{Y}]\le\frac{1}{2}\varepsilon[X]+\frac{1}{2}\varepsilon[Y],\quad \varepsilon[X+Y]\le\varepsilon[X]+\varepsilon[Y]$ is established. Therefore, the definition of sub-linear expectation can also be given according to conditions A1, A2, A4, A6. The same principle can be the following theorem.

Theorem 3 In a nonlinear expected space, if the nonlinear mathematical expectation satisfies the super-additive and homogeneity, then the expectation satisfies the concaveness. Principles and conclusions as above.

Theorem 4 That real-valued functional $\varepsilon:L(\Lambda,F,P)\to R$ are sub-linear mathematical expectations if they satisfy the following properties.

(1) Monotony: $X\ge Y,\varepsilon[X]\ge\varepsilon[Y]$. (2) Sub-additive: $\varepsilon[X+Y]\le\varepsilon[X]+\varepsilon[Y]$.

(3) Homogeneous: $\varepsilon[\lambda X]=\lambda\varepsilon[X],\forall\lambda\ge0.$ (4) (positive) translation invariant:

$\varepsilon(X+a)=\varepsilon(X)+a.$

Proof Due to the traditional sub-linear expectation to meet Monotony, sub-additive, homogeneous, constant property. The properties of the first three are the same, so only the permutation needs to be deduced here. The following derivation is made by using homogeneous and translation invariant.

When $X=a,\varepsilon[a+a]=\varepsilon[2a]=2\varepsilon[a]=\varepsilon[a]+a,\varepsilon[a]=a.$

Therefore, in the sub-additive expectation and the sub-linear expectation, the permutation can be

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replaced by (positive) translation invariant. Similarly, in the case of super-additive expectations and super-linear expectations, permutation can also be replaced by (positive) translation invariant.

Theorem 5 $\hat{\varepsilon}[X]=-\varepsilon[-X], \varepsilon[X]$ is a sub-linear expectation, $\hat{\varepsilon}[X]$ is a conjugate sub-linear expectation. There $\hat{\varepsilon}[X]$ is a super-linear expectation.

Proof (1) Monotony: $X \geq Y, -X \leq -Y, \varepsilon[-X] \leq \varepsilon[-Y], -\varepsilon[-X] \geq -\varepsilon[-Y]$, $\hat{\varepsilon}[X] \geq \hat{\varepsilon}[Y]$.

(2) Permutation: $\hat{\varepsilon}[c]=-\varepsilon(-c)=-(-c)=c, \hat{\varepsilon}[c]=c.$

(3) Homogeneous: $\hat{\varepsilon}[\lambda X]=-\varepsilon(-\lambda X)=\lambda[-\varepsilon(-X)]=\lambda \hat{\varepsilon}[X], \lambda>0.$

(4) Super-additive: $\hat{\varepsilon}[X+Y]=-\varepsilon(-X-Y)$, $\varepsilon(-X-Y) \leq \varepsilon(-X)+\varepsilon(-Y)$,

$-\varepsilon(-X-Y) \geq -\varepsilon(-X)-\varepsilon(-Y)=\hat{\varepsilon}[X]+\hat{\varepsilon}[Y]$, that is $\hat{\varepsilon}[X+Y] \geq \hat{\varepsilon}[X]+\hat{\varepsilon}[Y].$

The conjugate of sub-linear expectations, conjugates the conjugation of the conjugation of the quadratic expectation, and the conjugate sub-linear expectation is a super-linear expectation, as evidenced.

Similarly, there can be a conjugate of a super-linear expectation, which is a sub-linear expectation. If only the monotony, permutation, and super-additive are verified to have immediate conjugate sub-additive expectations, it is a super-additive expectation. In the same way there is a conjugate super-plus expectation that is a secondary plus expectation.

Theorem 6 $\hat{\varepsilon}[X]=-\varepsilon[-X], \varepsilon[X]$ is a convex expectation, $\hat{\varepsilon}[X]$ is a conjugate linear expectation, then $\hat{\varepsilon}[X]$ is a concave expectation.

Proof It is proved that the constant and monotony have been verified in theorem 5, and the concave is verified below. Sub-additive $\varepsilon[\lambda X+(1-\lambda) Y] \leq \lambda \varepsilon[X]+(1-\lambda) \varepsilon[Y]$,

$\hat{\varepsilon}[\lambda X+(1-\lambda) Y]=-\varepsilon[\lambda(-X)+(1-\lambda)(-Y)]$,

$\varepsilon[\lambda(-X)+(1-\lambda)(-Y)] \leq \lambda \varepsilon[-X]+(1-\lambda) \varepsilon(-Y), -\varepsilon[\lambda(-X)+(1-\lambda)(-Y)]$

$\geq \lambda[-\varepsilon(-X)]+(1-\lambda)[-\varepsilon(-Y)]=\lambda \hat{\varepsilon}[X]+(1-\lambda) \hat{\varepsilon}[Y]$, $\hat{\varepsilon}[\lambda X+(1-\lambda) Y] \geq \lambda \hat{\varepsilon}[X]$

$+(1-\lambda) \hat{\varepsilon}(Y)$ is established for concave, conjugate convex expectation is a concave expectation. Vice versa, the conjugate expectation is a convex expectation.

In addition, there are many other nonlinear mathematical expectations, such as g-expectations, G-expectations, distorted mathematical expectations, entropy expectations, etc, which are not covered in this article, and can be learned more about the problem of nonlinear mathematical expectations by reading references [17-18] etc.

4. The result of the correspondence between the risk measure and the mathematical expectation and its proof

Any good risk measure needs to satisfy the sub-additive, and the establishment of this property excludes the possibility of additional risk after the merger of risk positions, and convexity has a similar nature. Therefore, there is sub-additive in the definition of consistent risk measures, and convexity appears in convexity risk measures. For risk measurement, Super-additive and concaveness will be detrimental to risk control, and it is not meaningful to discuss these two properties in the definition of risk measurement.

The development of risk measurement has undergone a process of continuous development. From the earlier use of the sub-additive characteristics of variance to use variance to represent risk measures, to the later introduction of consistency to define the consistent risk measure, and further the introduction of convexity to define the convex risk measure. In fact, using the quasi-convexity $\rho[\lambda X+\left(1-\lambda\right)\mathrm{Y}]$ $\leq \max(\lambda\rho[X]+(1-\lambda)\rho[Y]), X,Y \in L\left(\Lambda,F,P\right), \lambda \in \left[0,1\right]$, the pseudo-convex risk measure can be defined, and the distortion risk measurement established by using the distortion function $\rho_g(x)=Ex^*=\int x^* dF^*=\int_0^\infty g\left(F_x(t)\right)dt-\int_{-\infty}^0 (1-g\left(\overline{F_x}(t)\right))dt$ is also widely used. The introduction of the relative entropy risk measurement established by the concept of relative entropy is also an important convex risk measure[19]. The introduction of the concept of time flow in modern finance has transitioned the research in the field of traditional static risk measurement to the field of dynamic risk measurement[20].In this paper, the relationship between nonlinear mathematical expectations and some static risk measurements is mainly discussed, and the risk measurement involved is still based on the traditional consistent risk measurement and convex risk measurement. The following theorem is mainly derived.

Theorem 7 Lets ε be real functional on $L\left(\Lambda,F,P\right)$ space, $\rho\left(X\right)=-\varepsilon[X]$, $X \in L\left(\Lambda,F,P\right)$ and ρ is a risk measure when ε is a linear expectation.

Proof verification of the two characteristics of risk measurement.

$X \leq Y, \varepsilon[X] \leq \varepsilon[Y], -\varepsilon[X] \geq -\varepsilon[Y], \rho[X] \geq \rho[Y]$, Monotony is confirmed.

$\rho(X+M)=-\varepsilon(X+M)=-\varepsilon(X)-\varepsilon(M)=\rho(X)-M$, Translation invariant is confirmed.

Note 2: The above conclusion can also be drawn when $\rho(X)=\varepsilon\left(-X\right)$.

The theorem does not hold the opposite, linearity in linear expectations requires many conditions to be derived, and conditions in risk measurements cannot derive linearity. This shows that the risk measurement contains more information and the fact is the same.

Theorem 8 Lets ε be a real-valued functional in $L\left(\Lambda,F,P\right)$ space, $X \in L\left(\Lambda,F,P\right)$, so that $\rho\left(X\right)=\varepsilon[-X]$ and if ρ is a risk measure, then ε is a nonlinear mathematical expectation.

Proof there is $\varepsilon[X]=\rho\left(-X\right)$.

(1) $\varepsilon[c]=\rho\left(-c\right)=\rho\left(-c\right)-0=\rho\left(-c+0\right)=\rho\left(0\right)+c$, $\rho\left(0\right)=0$ is the initial value of the risk measure, and $\varepsilon\left[c\right]=c$, the perverse is established.

(2) $X \leq Y, -X \geq -Y, \rho(-X) \leq \rho(-Y), \varepsilon(X) \leq \varepsilon(Y)$. Monotony is established, proven. This theorem, on the contrary, cannot be sustained and cannot be used to derive translation invariant in financial risk measures using the nature of nonlinear mathematical expectations.

Theorem 9 ε is a real-valued functional in $L\left(\Lambda,F,P\right)$ space, such that $\rho\left(X\right)=\varepsilon[-X]$, $X \in L\left(\Lambda,F,P\right)$. If ρ is a consistent risk measure, ε is a Sub-additive expectation.

Proof $\varepsilon[X]=\rho\left(-X\right)$,

(1) Permutation: $\varepsilon[c]=\rho\left(-c\right)=\rho\left(-c\right)-0=\rho\left(-c+0\right)=\rho\left(0\right)+c=c$.

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(2) Monotony: $X \le Y, -X \ge -Y, \rho(-X) \le \rho(-Y), \varepsilon(X) \le \varepsilon(Y).$

(3) Sub-additive: $\varepsilon[X+Y] = \rho(-X-Y) \le \rho(-X) + \rho(-Y) = \varepsilon(X) + \varepsilon(Y)$.

But vice versa cannot have a corresponding conclusion. Because the consistent risk measure has a homogeneous time, the secondary plus expectation cannot be derived from the known conditions. Therefore, after adding the homogeneous sex, there are the following conclusions.

Let ε be a real-valued functional on $L(\Lambda, F, P)$ space, so that $\rho(X) = \varepsilon[-X]$, $X \in L(\Lambda, F, P)$. Then the following statements are equivalent to (i) ε being the sublinear expectation and (ii) ρ being the consistent risk measure.[16] In fact, if pure Monotony is not taken into account, the mathematical last linear expectation and consistent risk measure are themselves equivalence relations.

Theorem 10 Lets ε be a real-valued functional on $L(\Lambda, F, P)$ space, such that $\rho(X) = -\varepsilon[X],\ X \in L(\Lambda, F, P)$, then ρ is a consistent risk measure, and the ε is a super-additive mathematical expectation.

The argument of theorem 10 is similar to the argument of theorem 9 and is not discussed here. Similarly, after the super-additive mathematical expectation plus the homogeneous sex, the following conclusions can be reached.

Let ε be real-valued functional on $L(\Lambda, F, P)$ space, such that $\rho(X) = -\varepsilon[X]$, $X \in L(\Lambda, F, P)$ (i) ε is a super-linear expectation and (ii) ρ is a measure of consistent risk. [16]

Theorem 11 Lets ε be a real-valued functional on $L(\Lambda, F, P)$ space, such that $\rho(X) = \varepsilon[-X], X \in L(\Lambda, F, P)$,when the ε is convex, add any of the following conditions, to prove that ρ is a convex risk measure. (i.) sub-additive: $\varepsilon[X+c] \le \varepsilon[X] + c$.(ii.) homogeneous: $\varepsilon[\beta X] = \beta\varepsilon[X], \forall \beta > 0.$ (iii.) if $\varepsilon[(1-\lambda)X + c] \to \varepsilon[X+c]$, where $\lambda \to 0$, but $\lambda \ne 0$.

Proof The relevant conditions for validating the convex risk measure.

(1) Monotony: $X \ge Y, -X \le -Y, \varepsilon(-X) \le \varepsilon(-Y), \rho(X) \ge \rho(Y).$

(2) Convexity: $\rho[\lambda X + (1-\lambda)\mathrm{Y}] = \varepsilon[-\lambda X - (1-\lambda)\mathrm{Y}] \le \lambda\varepsilon[-X] + (1-\lambda)\varepsilon[-Y]$

$= \lambda\rho(X) + (1-\lambda)\rho(\mathrm{Y})$.

(3) Standardization: $\rho(0) = \varepsilon(-0) = \varepsilon(0) = 0$.

(4) The main argument is the establishment of translation invariant, that is,

$\rho(X+c) = \rho(X) - c, \forall c \in R.$ is true. It is known that $\rho(X+c) = \varepsilon(-X-c)$ is established.

If (i.) is true, then there is $\varepsilon(-X-c) \le \varepsilon(-X) + \varepsilon(-c) = \rho(X) - c$, on the other hand $\varepsilon(-X) = \varepsilon(-X-c) = \varepsilon(-X-c+c) \le \varepsilon(-X-c) + \varepsilon(c) = \varepsilon(-X-c) + c$.There is

$\varepsilon(-X-c) \ge \varepsilon(-X) - c = \rho(X) - c$. That is $\rho(X+c) = \rho(X) - c$ as evidenced.

If (ii.) is true, under convex expectation conditions, the homogeneity is set at the same time, according to the note of theorem 2, meaning that the sub-linearity is also true at this point. It can also be demonstrated by the (i.) method above.

If (iii.) is true, it is argued by convexity and permutation $\varepsilon[(1-\lambda)X + c] =$

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$\varepsilon[(1-\lambda)X+\lambda\frac{c}{\lambda}] \le (1-\lambda)\varepsilon[X]+\lambda\varepsilon(\frac{c}{\lambda}) = (1-\lambda)\varepsilon[X]+c, \lambda \to 0$ then there is

$\varepsilon[X+c] \le \varepsilon[X]+c$, where $X \in L(\Lambda, F, P), c \in R$ is then $\varepsilon[X+c] \le \varepsilon[X]+c$

satisfying the sub-additive. As can be seen from the derivation in (i.), its conclusion is true. The theorem is proven.

The condition added in reference [16] $\varepsilon[\left(1-\frac{1}{n}\right)X+c] \to \varepsilon[X+c]$,which is demonstrated by using the desired convexity and permutation, in fact

$\varepsilon[(1-\lambda)X+c] \to \varepsilon[X+c]$ is only a special case of the condition of (iii.) of this theorem.

Theorem 12 Lets the ε be a real-valued functional on $L(\Lambda, F, P)$ space, such that $\rho(X) = -\varepsilon[X],\ X \in L(\Lambda, F, P)$, when the ε is concave, add any of the following conditions, all of which can prove that ρ is a convex risk measure.

(i.) Super-additive: $\varepsilon[X+c] \ge \varepsilon[X]+c$,(ii.) Homogeneity: $\varepsilon[\beta X] = \beta\varepsilon[X], \forall \beta > 0.$

(iii.) If $\varepsilon[(1-\lambda)X+c] \to \varepsilon[X+c]$ where $\lambda \to 0$, but $\lambda \ne 0$.

The argument for this theorem is similar to the theorem 11 and will not be repeated here.

Of course, when the construct $\rho(X) = -\varepsilon[X],\ \rho$ are convex risk measures, then the ε is convex expectation. The constructed $\rho(X) = -\varepsilon[X]$, when ρ is a convex risk measure, the ε is a concave expectation.

In fact, the risk measurement generally has to meet the translation invariant, and the expectation generally does not have this requirement, which is also one of the differences between the two. Of course, the correspondence between risk measurement and mathematical expectation is not limited to the scope discussed in this article, and there should be some similar correspondence between expectations such as g expectation, entropy expectation, distorted mathematical expectation, etc. and dynamic risk measurement in the corresponding field, waiting for further discovery and research.

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