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Distribution of Index Parity in Zeckendorf Decompositions

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Abstract: *Every positive integer has a unique Zeckendorf decomposition as a sum of nonconsecutive Fibonacci numbers. Instead of counting all summands equally, we study the number of odd-indexed summands minus the number of even-indexed summands. For integers chosen uniformly from a half-open interval between consecutive Fibonacci numbers, we derive an exact recurrence and closed form for the generating polynomial of the resulting parity difference. The two-step recurrence combines the alternating index weights. It gives exact formulas for the mean, shows that the variance equals the interval index divided by the square root of five up to a bounded error, and yields a central limit theorem after centering and scaling.*

Keywords: *Zeckendorf decomposition; Fibonacci numbers; generating functions; recurrence relations; central limit theorem; parity difference*

1. Introduction

Zeckendorf decompositions provide a way to study the arithmetic structure of integers through the Fibonacci sequence. Zeckendorf's theorem states that every positive integer can be written as a sum of distinct, non-adjacent Fibonacci numbers.

A classical question asks how many summands occur in a typical decomposition. If $K_n(N)$ is the number of summands and N is chosen uniformly from $[F_n, F_{n+1})$, then Lekkerkerker's theorem [1] gives

$$\mathbb{E}[K_n(N)] = \frac{n}{\phi^2+1} + O(1), \qquad \phi = \frac{1+\sqrt{5}}{2}. \tag{1}$$

After centering and normalization, $K_n(N)$ has a Gaussian limiting distribution; see [1]. (Changing between the two standard indexing conventions for the Fibonacci sequence only changes the $O(1)$ term in (1).)

The statistic $K_n(N)$, however, records only the total number of summands. It does not describe where those summands occur in the Fibonacci sequence. Two decompositions can have the same number of summands but different numbers of odd-indexed and even-indexed terms.

Let $O_n(N)$ and $E_n(N)$ denote the numbers of odd-indexed and even-indexed summands, respectively. We study their difference

$$D_n(N) = O_n(N) - E_n(N),$$

in contrast with the total count

$$K_n(N) = O_n(N) + E_n(N).$$

For N chosen uniformly from

$$I_n = \{F_n, F_n + 1, \dots, F_{n+1} - 1\},$$

define the Laurent polynomial

$$P_n(t) = \sum_{N=F_n}^{F_{n+1}-1} t^{D_n(N)}. \tag{2}$$

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The coefficient of t^d in $P_n(t)$ is exactly the number of integers in I_n for which $D_n(N) = d$. Thus $P_n(t)/F_{n-1}$ is the probability generating function. Our main result is an explicit rational generating function for the sequence $P_n(t)$, separated according to the parity of n. From it we also obtain the limiting behavior of D_n.

2. Background and Notation

2.1 The Fibonacci sequence

We use the convention

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad (n \geq 3).$$

Thus the sequence begins 1,1,2,3,5,8,13, ... We will use

$$F_{n+1} - F_n = F_{n-1} \qquad \text{and} \qquad \phi = \frac{1+\sqrt{5}}{2}.$$

2.2 Zeckendorf decompositions

A Zeckendorf decomposition of a positive integer N is a representation

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_k},$$

where

$$i_1 > i_2 > \cdots > i_k \geq 2 \qquad \text{and} \qquad i_j - i_{j+1} \geq 2.$$

The restriction $i_k \geq 2$ removes the duplicate copy $F_1 = F_2 = 1$. Zeckendorf's theorem states that this representation exists and is unique [2, 3]. For example,

$$100 = 89 + 8 + 3 = F_{11} + F_6 + F_4.$$

If $N \in I_n$, define

$$O_n(N) = \#\{j : i_j \text{is odd}\}, \qquad E_n(N) = \#\{j : i_j \text{is even}\}.$$

Then $K_n = O_n + E_n$ and $D_n = O_n - E_n$.

2.3 The Fibonacci interval

For $n \geq 2$, let

$$I_n = \{F_n, F_n + 1, \ldots, F_{n+1} - 1\}.$$

Since $F_{n+1} - F_n = F_{n-1}$, the interval I_n contains exactly F_{n-1} integers. In particular,

$$P_n(1) = F_{n-1}. \tag{3}$$

3. Exact generating functions

For each index j, set

$$\varepsilon_j = \begin{cases} +1, & j\text{odd}, \\ -1, & j\text{even}. \end{cases}$$

Thus the contribution of F_j to D_n is ε_j.

Lemma 3.1 (Reduction to legal index sets). *For every $n \geq 2$, subtracting F_n gives a bijection between I_n and the legal subsets of $\{2,3,\ldots,n-2\}$, where legal means that no two selected indices are consecutive. For $n = 2$ or 3, this index set is understood to be empty.*

Proof. The case $n = 2$ is immediate. For $n \geq 3$, every $N \in I_n$ must contain F_n in its Zeckendorf decomposition. Indeed, the largest legal sum not using F_n is $F_{n-1} + F_{n-3} + \cdots = F_n - 1$. Since F_{n-1} cannot occur together with F_n, all remaining indices are at most $n - 2$.

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Conversely, write $N = F_n + M$, where $0 \le M < F_{n-1}$. The Zeckendorf decomposition of M uses only $F_2, \dots, F_{n-2}$: a term F_{n-1} would already be too large. Adding F_n preserves non-adjacency. The two constructions are inverse to one another, so the correspondence is a bijection.

Define

$$A_m(t) = \sum_S t^{\sum_{j \in S} \varepsilon_j}, \tag{4}$$

where the sum is over all legal subsets $S \subseteq \{2, \dots, m\}$. We use the initial values $A_0(t) = A_1(t) = 1$, corresponding to the empty set. Lemma 3.1 immediately gives

$$P_n(t) = t^{\varepsilon_n} A_{n-2}(t). \tag{5}$$

Proposition 3.2 (One-step recurrence). *For $m \ge 2$,*

$$A_m(t) = A_{m-1}(t) + t^{\varepsilon_m} A_{m-2}(t). \tag{6}$$

Proof. Partition the legal subsets of $\{2, \dots, m\}$ into two classes. If m is not selected, the contribution is $A_{m-1}(t)$. If m is selected, then $m-1$ cannot be selected; removing m leaves an arbitrary legal subset of $\{2, \dots, m-2\}$, and selecting m contributes the factor t^{ε_m}. Adding the two classes proves (6).

The coefficients in (6) alternate between t and t^{-1}. Combining two consecutive steps removes this alternation. Put

$$\tau(t) = 1 + t + t^{-1}. \tag{7}$$

If $w = t^{\varepsilon_m}$, then $t^{\varepsilon_{m-1}} = w^{-1}$ and $t^{\varepsilon_{m-2}} = w$. Therefore

$$\begin{aligned} A_m &= A_{m-1} + wA_{m-2} \\ &= A_{m-2} + w^{-1}A_{m-3} + wA_{m-2} \\ &= (1 + w + w^{-1})A_{m-2} - A_{m-4} \\ &= \tau(t)A_{m-2} - A_{m-4}. \end{aligned}$$

Multiplying by t^{ε_n} in (5) gives the same recurrence for P_n.

Theorem 3.3 (Exact recurrence and generating functions). *For $n \ge 6$,*

$$P_n(t) = \tau(t)P_{n-2}(t) - P_{n-4}(t), \tag{8}$$

with

$$P_2(t) = t^{-1}, \qquad P_3(t) = t,$$
$$P_4(t) = t^{-1} + t^{-2}, \qquad P_5(t) = 1 + t + t^2.$$

Equivalently, the ordinary generating functions in the interval index are

$$\sum_{k \ge 1} P_{2k}(t) z^{k-1} = \frac{t^{-1} - z}{1 - \tau(t)z + z^2}, \tag{9}$$

$$\sum_{k \ge 1} P_{2k+1}(t) z^{k-1} = \frac{t}{1 - \tau(t)z + z^2}. \tag{10}$$

Proof. The recurrence was derived above. The four initial polynomials follow by listing the legal subsets in Lemma 3.1. For a sequence R_k satisfying $R_k = \tau R_{k-1} - R_{k-2}$, multiplication by the denominator $1 - \tau z + z^2$ leaves only the two initial terms. Applying this once to the even subsequence and once to the odd subsequence gives (9) and (10). For background on rational generating functions and linear recurrences, see [4].

There is also a useful closed form. Define $Q_{-1} = 0$, $Q_0 = 1$, and

$$Q_r(t) = \tau(t)Q_{r-1}(t) - Q_{r-2}(t) \qquad (r \ge 1). \tag{11}$$

Extracting coefficients from (9)–(10) gives

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$$\begin{aligned} P_{2k+1}(t) &= tQ_{k-1}(t), \\ P_{2k}(t) &= t^{-1}Q_{k-1}(t) - Q_{k-2}(t) \end{aligned} \qquad (k \geq 1). \tag{12}$$

If

$$\lambda_{\pm}(t) = \frac{\tau(t) \pm \sqrt{\tau(t)^2 - 4}}{2}, \tag{13}$$

then $\lambda_+\lambda_- = 1$ and

$$Q_r(t) = \frac{\lambda_+(t)^{r+1} - \lambda_-(t)^{r+1}}{\lambda_+(t) - \lambda_-(t)}. \tag{14}$$

Although (14) contains a square root, the expression simplifies to the Laurent polynomial determined by (11).

As a quick check, setting $t = 1$ makes $\tau = 3$. Equations (8) and (3) then reduce to the corresponding two-step Fibonacci identity, so the total number of represented integers is correct.

4. Mean, variance, and limiting distribution

Let

$$M_n(s) = \mathbb{E}[e^{sD_n}] = \frac{P_n(e^s)}{P_n(1)}$$

be the moment generating function. The exact formulas above already show a parity effect in the mean.

Proposition 4.1 (Exact mean). *For $n \geq 2$,*

$$\mathbb{E}[D_n] = \begin{cases} 1, & n\text{odd}, \\ -\dfrac{F_n}{F_{n-1}}, & n\text{even}. \end{cases} \tag{15}$$

Consequently, the mean is 1 along odd intervals and tends to $-\phi$ along even intervals.

Proof. Since $\tau'(1) = 0$, every $Q_r(t)$, viewed as a polynomial in $\tau(t)$, also has derivative zero at $t = 1$. For $n = 2k + 1$, the first formula in (12) therefore gives $P'_{2k+1}(1) = P_{2k+1}(1)$. Hence the mean is 1.

For $n = 2k$, only the derivative of t^{-1} contributes at $t = 1$, so

$$P'_{2k}(1) = -Q_{k-1}(1).$$

The recurrence $Q_r(1) = 3Q_{r-1}(1) - Q_{r-2}(1)$, with $Q_0(1) = 1$, shows that $Q_{k-1}(1) = F_{2k}$. Dividing by $P_{2k}(1) = F_{2k-1}$ proves (15).

Theorem 4.2 (Asymptotic variance and central limit theorem). *Let $\mu_n = \mathbb{E}[D_n]$ and $\sigma_n^2 = \mathrm{Var}(D_n)$. Then*

$$\sigma_n^2 = \frac{n}{\sqrt{5}} + O(1). \tag{16}$$

Moreover,

$$\frac{D_n - \mu_n}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \qquad (n \to \infty). \tag{17}$$

Proof. Write $t = e^s$. Then

$$\tau(e^s) = 1 + e^s + e^{-s} = 1 + 2\cosh s.$$

Let $\lambda(s)$ be the larger root in (13). Near $s = 0$, $\lambda(s)$ is analytic, $\lambda(0) = \phi^2$, and the other root is $\lambda(s)^{-1}$. The closed forms (12) and (14) imply, for either parity of n,

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$$\log M_n(s) = \frac{n}{2}\big(\log\lambda(s) - \log\lambda(0)\big) + B_n(s), \tag{18}$$

where $B_n(s)$ and its derivatives remain bounded for s in a fixed neighborhood of 0. This bounded term accounts for the endpoint and parity factors in (12).

The defining equation for the dominant root is

$$\lambda(s) + \lambda(s)^{-1} = 1 + 2\cosh s.$$

Its first derivative at 0 is zero. Differentiating twice and using $\lambda(0) = \phi^2$ gives

$$\frac{d^2}{ds^2}\log\lambda(s)\Big|_{s=0} = \frac{2}{\lambda(0) - \lambda(0)^{-1}} = \frac{2}{\sqrt{5}}.$$

Taking the second derivative of (18) proves (16).

Finally, $\log\lambda(s)$ is even and analytic near 0, so

$$\log\lambda(s) - \log\lambda(0) = \frac{s^2}{\sqrt{5}} + O(s^4).$$

Set $s = iu/\sigma_n$ and subtract the exact linear term $s\mu_n$ from (18). The centered contribution from $B_n(s)$ is $O(s^2)$, hence tends to zero. Using (16), the logarithm of the resulting characteristic function tends to $-u^2/2$ for each fixed real u. Thus the characteristic functions converge to that of a standard normal random variable, which proves (17). For the general quasi-powers approach to combinatorial central limit theorems, see [5].

Remark 4.3. For odd $n = 2k + 1$, $P_n(t) = tQ_{k-1}(t)$ and Q_{k-1} depends on t only through $1 + t + t^{-1}$. Hence the distribution is symmetric about $D_n = 1$. The even case has a different boundary term, which explains the two different means in Proposition 4.1. This parity effect stays bounded, while the standard deviation grows on the order of $\sqrt{n}$.

5. A small example

Take $n = 6$. The interval is $I_6 = \{8,9,10,11,12\}$, and the decompositions are shown below.

N	Decomposition	Indices	$D_6(N)$
8	F_6	6	-1
9	$F_6 + F_2$	6, 2	-2
10	$F_6 + F_3$	6, 3	0
11	$F_6 + F_4$	6, 4	-2
12	$F_6 + F_4 + F_2$	6, 4, 2	-3

Therefore

$$P_6(t) = 1 + t^{-1} + 2t^{-2} + t^{-3}.$$

This agrees with Theorem 3.3, since $P_6 = \tau P_4 - P_2$, and $P_6(1) = 5 = F_5$.

6. Conclusion

The parity difference is governed by a particularly simple two-step recurrence. The factor $1 + t + t^{-1}$ records the three possibilities that appear after one odd and one even index are grouped together. The exact generating functions show that the mean has a bounded parity-dependent shift, whereas the variance grows linearly. At the scale of the standard deviation, the boundary effect disappears and the centered statistic approaches a Gaussian distribution.

The same method can be extended by replacing the two alternating weights with periodic weights of a longer period. The resulting transfer matrix is larger, but its dominant eigenvalue again controls the mean, variance, and possible limiting law.

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References

[1] M. Kologlu, G. S. Kopp, S. J. Miller, and Y. Wang, On the number of summands in Zeckendorf decompositions, The Fibonacci Quarterly, vol. 49, no. 2, pp. 116–130, 2011.
[2] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, Bulletin de la Société Royale des Sciences de Liège, vol. 41, pp. 179–182, 1972.
[3] J. L. Brown, Jr., Zeckendorf's theorem and some applications, The Fibonacci Quarterly, vol. 2, no. 3, pp. 163–168, 1964.
[4] P. Flajolet and R. Sedgewick, Analytic Combinatorics, Cambridge University Press, Cambridge, 2009.
[5] H.-K. Hwang, On convergence rates in the central limit theorems for combinatorial structures, European Journal of Combinatorics, vol. 19, no. 3, pp. 329–343, 1998.