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Center-focus identification of quasi-homogeneous polynomial planar rigid system

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Abstract: *For each quasi-homogeneous polynomial planar rigid system with weight (2,1), we prove that the origin is a center equilibrium when the degree is odd, and we obtain necessary and sufficient condition for the origin to be a center when the degree is even.*

Keywords: *quasi-homogeneous polynomial, planar rigid system, center-focus identification, symmetry principle*

1. Introduction

A planar differential system is called a *rigid system* ([4, 7, 10]) if its angular speed is constant. It is proved in [11] that each planar polynomial rigid system can be transformed by a non-degenerate linear transformation together with a time rescaling into the following form

$$\begin{cases} \dfrac{dx}{dt} = -y + xF(x,y), \\ \dfrac{dy}{dt} = x + yF(x,y), \end{cases} \tag{1}$$

Where $F(x,y)$ is a polynomial and $F(0,0) = 0$. In the polar coordinates $x = rcos\theta$ and $y = rsin\theta$, system (1) becomes

$$\frac{dr}{dt} = rF(rcos\theta, rsin\theta), \qquad \frac{d\theta}{dt} = 1.$$

It follows that the origin is the only equilibrium of system (1) and if it is a center then it is a uniformly isochronous center [11], i.e., the center-focus problem of system (1) is equivalent to the isochronicity problem.

So far, the center-focus problem of system (1) has attracted the attention of many authors. In [11], the author considered the case $F(x,y) = H_p(x,y)$, a homogeneous polynomial of degree $p \geq 0$, and proved that the origin is always a center when p is odd and the origin is a center if and only if the system is time reversible when p is even. It is proved in [5] that the origin is a center of system (1) with $F(x,y) = H_1(x,y) + H_2(x,y)$ if and only if the system is time reversible. The same results are also obtained in [2, 3] for system (1) with $F(x,y) = H_1(x,y) + H_p(x,y)$, $F(x,y) = H_2(x,y) + H_{2p}(x,y)$ and $F(x,y) = H_1(x,y) + H_2(x,y) + H_3(x,y) + H_4(x,y)$. Authors in [1, 4, 6, 9] investigated center-focus problem of system (1) in the case $F(x,y) = H_0(x,y) + H_p(x,y) + H_q(x,y)$. In particular, authors in [8] obtained the center conditions in the case $F(x,y) = H_p(x,y) + H_{2p}(x,y)$ for $p = 2, 3, 4$ and 5. Moreover, a separable polynomial case, i.e., $F(x,y) = f(x)g(y)$ for some polynomials $f(x)$ and $g(y)$, is considered in [7].

A polynomial $P(x,y)$ is referred to as a *quasi-homogeneous polynomial of degree* n *with weight* (s_1, s_2) if s_1 and s_2 are positive coprime integers and $P(\lambda^{s_1}x, \lambda^{s_2}y) = \lambda^n P(x,y)$. We call system (1) a quasi-homogeneous polynomial planar rigid system of degree n with weight (s_1, s_2) if the polynomial $F(x,y)$ given in (1) is a quasi-homogeneous polynomial of degree n with weight (s_1, s_2). Many authors considered the center-focus problem of quasi-homogeneous polynomial differential equations, see [12, 13] for example. However, as far as we known, there are no results concerning about the center-focus problem of quasi-homogeneous polynomial rigid system (1).

In this paper, we consider a quasi-homogeneous polynomial planar rigid system of degree $n(\geq$

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1) with weight $(2,1)$, i.e. the following system

$$\begin{cases} \dfrac{dx}{dt} = X(x,y) := -y + xQ_n(x,y), \\ \dfrac{dy}{dt} = Y(x,y) := x + yQ_n(x,y), \end{cases} \tag{2}$$

where

$$Q_n(x,y) := \sum_{2i+j=n} \alpha_{i+j+1-\left[\frac{n+1}{2}\right]} x^i y^j$$

and $\left[\frac{n+1}{2}\right]$ denotes the largest integer being $\le \frac{n+1}{2}$.

2. Main results

We discuss the parity of degrees of quasi-homogeneous polynomials with weights $(2,1)$ separately. First consider the case of odd order, and mainly use the principle of symmetry to give the result of its center-focus distinction.

Theorem 1. Equilibrium $O:(0,0)$ of system (2) with odd n is a center.

Proof. Since n is odd, we assume that $n = 2k+1$ for an integer $k \ge 0$. Then $Q_n(x,y)$ given in (2) becomes

$$Q_{2k+1}(x,y) = \alpha_1 x^k y + \alpha_2 x^{k-1} y^3 + \cdots + \alpha_k x y^{2k-1} + \alpha_{k+1} y^{2k+1}, \tag{3}$$

an odd function in y. It follows from (2) that $X(x,y) = -X(x,-y)$ and $Y(x,y) = Y(x,-y)$, i.e., the vector field generated by system (2) is symmetric about the x-axis. By the symmetry principle given in [14], the equilibrium O of system (2) is a center. This completes the proof.

Theorem 2. Equilibrium $O:(0,0)$ of system (2) with even $n = 2k$ has the following properties: In the case that k is odd,

(ia) if $\alpha_2 = \alpha_4 = \cdots = \alpha_{2(s-1)} = 0$ and $\alpha_{2s} < 0$ (resp. > 0), then the equilibrium O is a stable (resp. unstable) weak focus of order $\frac{k-1}{2} + s$, where $s = 1,2,\ldots,\frac{k+1}{2}$;

(ib) if $\alpha_2 = \alpha_4 = \cdots = \alpha_{k+1} = 0$, then the equilibrium O is a center,

and in the case that k is even,

(iia) if $\alpha_1 = \alpha_3 = \cdots = \alpha_{2s-1} = 0$ and $\alpha_{2s+1} < 0$ (resp. > 0), then the equilibrium O is a stable (resp. unstable) weak focus of order $\frac{k}{2} + s$, where $s = 0,1,\ldots,\frac{k}{2}$;

(iib) if $\alpha_1 = \alpha_3 = \cdots = \alpha_{k+1} = 0$, then the equilibrium O is a center.

Proof. When $n = 2k$, the polynomial $Q_n(x,y)$ given in (2) becomes

$$Q_{2k}(x,y) = \alpha_1 x^k + \alpha_2 x^{k-1} y^2 + \cdots + \alpha_k x y^{2k-2} + \alpha_{k+1} y^{2k}. \tag{4}$$

Under polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we can rewrite system (2) as

$$\frac{dr}{d\theta} = rQ_{2k}(r\cos\theta, r\sin\theta) = \sum_{i=1}^{k+1} \alpha_i r^{k+i} \cos^{k+1-i}\theta \sin^{2i-2}\theta\,. \tag{5}$$

Let $r(\theta,c)$ be the solution of system (5) satisfying that $r(0,c) = c$. By the analytical dependence on initial conditions of solutions, $r(\theta,c)$ can be expanded as

$$r(\theta,c) = r_1(\theta)c + r_2(\theta)c^2 + r_3(\theta)c^3 + \cdots. \tag{6}$$

We see from the condition $r(0,c) = c$ that $r_1(0) = 1$ and $r_\ell(0) = 0$ for all $\ell \ge 2$. Substituting the power series (6) into equation (5), we obtain

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$$\frac{d}{d\theta}\left(\sum_{j=1}^{\infty} r_j(\theta)c^j\right)=\sum_{i=1}^{k+1}\alpha_i\left(\sum_{j=1}^{\infty} r_j(\theta)c^j\right)^{k+i} cos^{k+1-i}\theta\, sin^{2i-2}\theta$$

$$=\sum_{j=k+1}^{\infty}\sum_{i=1}^{\sigma}\left(\alpha_i cos^{k+1-i}\theta sin^{2i-2}\theta \sum_{\substack{\tau_1+\tau_2+\cdots+\tau_{k+i}=j\\ \tau_1,\tau_2,\ldots,\tau_{k+i}\geq 1}}\prod_{\ell=1}^{k+i} r_{\tau_\ell}(\theta)\right)c^j. \qquad (7)$$

Where $\sigma := min\{j-k, k+1\}$. Comparing the coefficients of the same degree of c in both sides of the above equation, we obtain differential equations

$$\frac{dr_1(\theta)}{d\theta}=\frac{dr_2(\theta)}{d\theta}=\cdots=\frac{dr_k(\theta)}{d\theta}=0. \qquad (8)$$

By the initial conditions given just below (6),

$$r_1(\theta)=1, \qquad r_2(\theta)=r_3(\theta)=\cdots=r_k(\theta)=0. \qquad (9)$$

On the other hand, $r_{k+1}(\theta), r_{k+2}(\theta), \ldots, r_{2k}(\theta)$ satisfy that

$$\frac{dr_{k+s}(\theta)}{d\theta}=\sum_{i=1}^{s}\alpha_i cos^{k+1-i}\theta sin^{2i-2}\theta \sum_{\substack{\tau_1+\tau_2+\cdots+\tau_{k+i}=k+s\\ \tau_1,\tau_2,\ldots,\tau_{k+i}\geq 1}}\prod_{\ell=1}^{k+i} r_{\tau_\ell}(\theta), \quad s=1,2,\cdots,k.$$

The above equations can be simplified by (9) as

$$\frac{dr_{k+s}(\theta)}{d\theta}=\alpha_s cos^{k+1-s}\theta sin^{2s-2}\theta, \qquad s=1,2,\ldots k. \qquad (10)$$

In order to compute the focal values, we need the following two integrals

$$\int_0^{2\pi} cos^{2p+1}\theta sin^{2q}\theta\, d\theta=0, \qquad \int_0^{2\pi} cos^{2p}\theta sin^{2q}\theta\, d\theta=2\pi I_{p,q},$$

Where p and q are nonnegative integers and

$$I_{p,q}:=\sum_{\ell=0}^{q}(-1)^\ell\binom{q}{\ell}\frac{(2(p+\ell)-1)!!}{\big(2(p+\ell)\big)!!}.$$

In the case that k is odd, we assume that $k=2m+1$ for an integer $m\geq 1$. Solving differential equations (10) with the initial conditions given just below (6) and integrals given just below (10), we obtain that

$$\begin{cases} r_{2m+1+s}(2\pi)=2\pi\alpha_s I_{m+1-\frac{1}{2}s,s-1}, & for\ even\ s\in\{1,2,\ldots,2m+1\},\\ r_{2m+1+s}(2\pi)=0, & for\ odd\ s\in\{1,2,\ldots,2m+1\}.\end{cases} \qquad (11)$$

Consequently, focal values are given by

$$\begin{cases} g_{2\rho+1}=0, & \rho=1,2,\ldots,m,\\ g_{2\rho+1}=\alpha_{2(\rho-m)+2}I_{2m-\rho,2(\rho-m)+1}, & \rho=m+1,m+2,\ldots,2m.\end{cases}$$

Thus result (ia) holds for all $s=1,2,\ldots,\frac{k-1}{2}$.

In order to show that result (ia) also holds for $s=\frac{k+1}{2}=m+1$, we need to compute the $(2m+1)$-th focal value, which leads to consider the following equation

$$\frac{dr_{4m+3}(\theta)}{d\theta}=\sum_{i=1}^{2m+2}\alpha_i cos^{2m+2-i}\theta sin^{2i-2}\theta \sum_{\substack{\tau_1+\tau_2+\cdots+\tau_{2m+1+i}=4m+3\\ \tau_1,\tau_2,\ldots,\tau_{2m+1+i}\geq 1}}\prod_{\ell=1}^{2m+1+i} r_{\tau_\ell}(\theta),$$

obtained from (7). We can further simplify the above equation by (9) as

$$\frac{dr_{4m+3}(\theta)}{d\theta}=(2m+2)\alpha_1 r_{2m+2}(\theta)cos^{2m+1}\theta+\alpha_{2m+2}sin^{4m+2}\theta. \qquad (12)$$

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Using the initial condition $r_{4m+3}(0) = 0$ given just below (6) and integrals given just below (12), we obtain the $(2m+1)$-th focal value

$$g_{4m+3} = \frac{1}{2\pi} r_{4m+3}(2\pi) = \alpha_{2m+2} \frac{(4m+1)!!}{(4m+2)!!}.$$

Thus, result (ia) also holds in the case $s = (k+1)/2$.

Next, we turn to prove (ib). If $\alpha_2 = \alpha_4 = \cdots = \alpha_{k+1} = 0$, then the polynomial (4) becomes

$$Q_{2k}(x,y) = \alpha_1 x^k + \alpha_3 x^{k-2} y^4 + \cdots + \alpha_{k-2} x^3 y^{2k-6} + \alpha_k x y^{2k-2}, \tag{13}$$

an odd function in x. We see from (2) that $X(x,y) = X(-x,y)$ and $Y(x,y) = -Y(-x,y)$, i.e., the vector field generated by system (2) is symmetric about the y-axis. By the symmetry principle given in [14], the equilibrium O is a center.

In the case that k is even, we assume that $k = 2m$ for an integer $m \geq 1$. Similarly to equalities (11) in the above case, we have

$$\begin{cases} r_{2m+s}(2\pi) = 0, & for\ even\ s \in \{1,2,\ldots,2m\}, \\ r_{2m+s}(2\pi) = 2\pi\alpha_s I_{m+\frac{1-s}{2},s-1}, & for\ odd\ s \in \{1,2,\ldots,2m\} \end{cases} \tag{14}$$

and therefore focal values are given by

$$\begin{cases} g_{2\rho+1} = 0, & \rho = 1,2,\ldots,m-1, \\ g_{2\rho+1} = \alpha_{2(\rho-m)+1} I_{2m-\rho,2(\rho-m)}, & \rho = m, m+1, \ldots, 2m-1. \end{cases} \tag{15}$$

Therefore result (iia) holds for $s = 0,1,\ldots,\frac{k}{2}-1$.

In order to show that result (iia) also holds for $s = \frac{k}{2} = m$, we need to compute the $2m$-th focal value. Similarly to (12), we consider the equation

$$\frac{dr_{4m+1}(\theta)}{d\theta} = (2m+1)\alpha_1 cos^{2m}\theta r_{2m+1}(\theta) + \alpha_{2m+1} sin^{4m}\theta.$$

By the assumption that $\alpha_1 = \alpha_3 = \cdots = \alpha_{2m-1} = 0$ given in (iia) and the initial condition $r_{4m+1}(0) = 0$ given just below (6), we obtain the $2m$-th focal value

$$g_{4m+1} = \frac{1}{2\pi} r_{4m+1}(2\pi) = \alpha_{2m+1} \frac{(4m-1)!!}{(4m)!!}.$$

Then result (iia) also holds in the case $s = \frac{k}{2}$.

Finally, the same as case (ib), the vector field generated by system (2) is also symmetric about the y-axis in case (iib), i.e., $\alpha_1 = \alpha_3 = \cdots = \alpha_{k+1} = 0$. Thus, the equilibrium O is a center by the symmetry principle given in [14] and therefore this theorem is proved.

3. Conclusion

In this paper, each quasi-homogeneous polynomial planar rigid system with weights (2,1) is studied. The results show that the equilibrium is the center when the degree n of the quasi-homogeneous polynomial with weights (2,1) is odd, and the sufficient and necessary condition that the origin is the center when the degree n is even.

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