ISSN 2616-5805 Vol. 5, Issue 3: 142-146, DOI: 10.25236/AJMS.2024.050316

The Limit and the Arithmetic Operations of Sequences in Split Complex Plane

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Abstract: *This paper studies arithmetic operations sequences of split complex numbers, which form a commutative ring with zero divisors based on two real numbers. These operations are fundamental to the theory of split complex limit. Addressing the challenge of zero divisors within split complex numbers, the paper leverages their decomposition properties to devise an algorithm for the convergent separation of sequences. This development is pivotal for advancing the limit theory on the split complex numbers plane. Furthermore, the research extends its implications to the application of physical direction, providing a solid groundwork for both theoretical exploration and practical application in physics and related fields. The paper's results not only enhance our understanding of the algebraic structure of split complex numbers but also open new avenues for their utilization in modeling physical phenomena.*

Keywords: *Arithmetic Operations, Split Complex Numbers, Zero Divisors*

1. Introduction

This paper studys the uncharted territories of mathematical analysis, introducing and expanding the concept of hyperbolic conformality for bicomplex functions. This concept, initially confined to the realm of complex variables, is now extended to the intriguing set of hyperbolic numbers, denoted as$\mathbb{D}^n$. This paper presents the foundational principles of analysis within$\mathbb{D}^n$, alongside a generalization of geometric hyperbolic objects that were once defined exclusively within the purview of bicomplex analysis [1].M. E. LunaElizarraras meticulously examines the integration of hyperbolicvalued functions, offering a novel approach that embraces the concept of partial order inherent to hyperbolic numbers and the structure of hyperbolic intervals [2]. This systematic study not only enriches our understanding of integration but also opens new avenues for exploration in the calculus of hyperbolic functions.Furthermore, this paper traverses the landscape of split complex analysis, revisiting the Abel theorem of power series and establishing new criteria for the convergence of these series. The paper provides a visual tapestry of convergence regions within the split complex space, offering a graphical insight into the behavior of these series [3].

In a foray into matrix exponentials, this paper scrutinizes the exponential functions of 2×2 split complex matrices, dissecting their properties across a spectrum of scenarios [4]. This multidimensional examination unveils fresh insights into the behavior of exponential functions within the complex domain.The discourse on linear functionals is elevated as this paper discusses their properties within the context of topological hyperbolic and topological bicomplex modules, revealing the subtle interplay between linearity and topology [5]. Golberg A culminate in an indepth analysis of bicomplex Mobius transformations, proving various algebraic and geometric results that harness the power of both hyperbolic and bicomplex geometric objects [6]. this paper extends this exploration to the Lorentzian plane, where this papers determines affine transformations using hyperbolic numbers and present a gallery of hyperbolic fractals [7].In a testament to the multidimensional nature of our study, this papers construct fractaltype sets in fourdimensional space, utilizing hyperbolic geometrical objects and the partial order on hyperbolic numbers, thus bridging the gap between fractal theory and hyperbolic geometry [8].The paper presents a detailed study of hyperbolic Mobius transformations in the bicomplex space $\mathbb{BC}$, isomorphic to$\mathbb{D}^n$, including an innovative conjugacy classification and a proof of the invariance of the crossratio under such transformations [9].

Based on the above research, this paper studies operation of limit in split complex plane. Since the split complex number has a zero factor, this paper overcomes the complexity of the zero factor. By separating the decomposition of the split complex number, this paper obtains limit and the arithmetic

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operations of sequences in split complex plane. It can be rephrased in English as follows: It establishes the foundational principles for essential theorems within the domain of the split complex plane, encompassing the Pinching Theorem, the Theorem of Uniform Convergence, the Cauchy's Test for Convergence, the Theorem on Bounded Sets, and the HeineBorel Theorem. Additionally, it enhances the integration of split complex numbers into the field of physics, offering a solid mathematical basis for both the theoretical examination and the practical application of these numbers.

2. Preliminaries

2.1 Split Complex Numbers

We already know that the real source of introducing complex numbers starts from the root solution of the cubic equation. So the definition of complex field is as follows:

$$\mathbb{C} = \{z = z_1 + z_2 i \mid x, y \in \mathbb{R}, i^2 = -1\} \tag{1}$$

From the above definition, we can know that the complex field $\mathbb{C}$ is generated by the real and imaginary i units.

Next, we give the concept of split complex numbers:

$$\mathbb{R}^{1,1} = \{\zeta = x + yj \mid x, y \in \mathbb{R}, j^2 = 1\} \tag{2}$$

It is not difficult to see that the separation of complex field and split complex numbers has strong similarities but they are also different, such as the 0 element generation or de-composition. If

$$0 = a \cdot b \tag{3}$$

In the complex plane we have$a = 0$or$b = 0$, in the split complex plane $\mathbb{R}^{1,1}$ we have

$$0 = (1 + j)(1 - j) = 1 - j^2 \tag{4}$$

In the split complex plane $\mathbb{R}^{1,1}$, the two nonzero factors can generate 0.

If we define

$$j_+ = \frac{(1+j)}{2} \quad \text{and} \quad j_- = \frac{(1-j)}{2}. \tag{5}$$

They constitute the entire ring as idempotent zero divisors, if$\zeta = x + yj \in \mathbb{R}^{1,1}$,then it can be expressed as follows:

$$\zeta = u j_+ + v j_- \tag{6}$$

The set $\mathbb{R}^{1,1}$ constitutes a commutative ring, endowed with the operations of addition and multiplication, which are defined as follows:

$$\begin{aligned} \zeta_1 + \zeta_2 &= (x_1 + y_1 j) + (x_2 + y_2 j) \\ &= (x_1 + x_2) + (y_1 + y_2) j \end{aligned} \tag{7}$$

$$\begin{aligned} \zeta_1 \cdot \zeta_2 &= (x_1 + y_1 j) \cdot (x_2 + y_2 j) \\ &= (x_1 x_2 + y_1 y_2) + (x_1 y_2 + x_2 y_1) j. \end{aligned} \tag{8}$$

The zero factor has the following algorithm

$$j_+ \cdot j_- = 0, \quad {j_-}^2 = j_-, \quad {j_+}^2 = j_+, \tag{9}$$

$$j_+ + j_- = 1, \quad j_+ - j_- = j. \tag{10}$$

Therefore, for $\zeta_1 = u_1 j_+ + v_1 j_-, \zeta_2 = u_2 j_+ + v_2 j_-$, we have

$$\begin{aligned} \zeta_1 \cdot \zeta_2 &= (u_1 j_+ + v_1 j_-) \cdot (u_2 j_+ + v_2 j_-) \\ &= u_1 u_2 (j_+)^2 + u_1 v_2 j_+ j_- + v_1 u_2 j_- j_+ + v_1 v_2 (j_-)^2 \\ &= u_1 u_2 j_+ + v_1 v_2 j_- \end{aligned} \tag{11}$$

Obviously this means that $\mathbb{R}^{1,1}$as an algebra, is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.So we have

$$\zeta = uj_{+} \text{or } \zeta = uj_{-} \tag{12}$$

2.2 Definition of the partial order

First, we define the positive split complex numbers as follows:

$$\mathbb{R}^{1,1}_{+} = \{\zeta = uj_{+} + vj_{-} \mid u, v \succcurlyeq 0\} \tag{13}$$

Therefore, the same can be obtained the negative split complex numbers,

$$\mathbb{R}^{1,1}_{-} = \{\zeta = uj_{+} + vj_{-} \mid u, v \preccurlyeq 0\} \tag{14}$$

$\mathbb{R}^{1,1}_{+}$,$\mathbb{R}^{1,1}_{-}$can be showed in the Figure 1

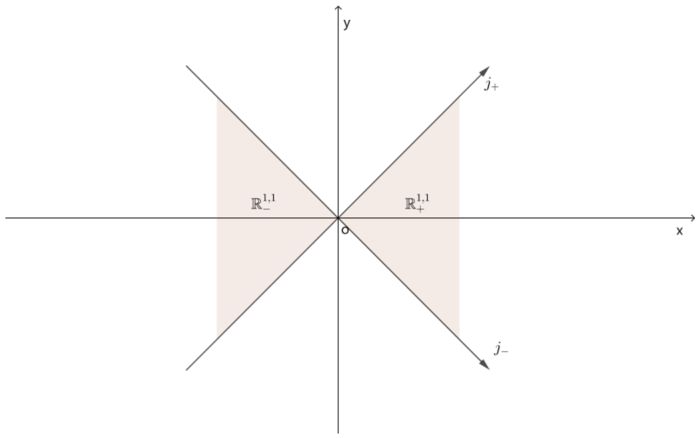

Figure 1: The positive and negative split complex numbers

The Figure 1 depicts two regions that bear a strong resemblance to the first and third quadrants of the real number plane, allowing for a more intuitive understanding of the connection between the following text and many properties within the domain of real numbers[10-12].

Based on the above definition, we give the definition of the comparison size,

$$\zeta_1 \succcurlyeq \zeta_2 \Leftrightarrow \zeta_1 - \zeta_2 \in \mathbb{R}^{1,1}_{+} \tag{15}$$

$$\zeta_1 \preccurlyeq \zeta_2 \Leftrightarrow \zeta_1 - \zeta_2 \in \mathbb{R}^{1,1}_{-}$$

$$\zeta_1, \zeta_2 \in \mathbb{R}^{1,1}. \tag{16}$$

So we lead to the concept of split complex interval:

$$[\zeta, \omega]_{\mathbb{R}^{1,1}} = \{u \in \mathbb{R}^{1,1} \mid \zeta \preccurlyeq u \preccurlyeq \omega\}, \tag{17}$$

for example Figure 2

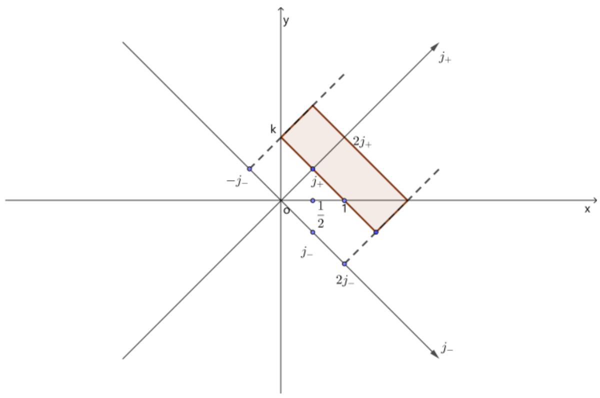

Figure 2: The split complex interval $[k, 2]_{\mathbb{R}^{1,1}}$

In the theory of metric spaces, the concept of intervals helps define limit points and accumulation points. If every neighborhood of a sequence contains infinitely many points of that sequence, then the

ISSN 2616-5805 Vol. 5, Issue 3: 142-146, DOI: 10.25236/AJMS.2024.050316

limit of the sequence is referred to as an accumulation point. In optimization and variational methods, the concept of intervals is used to define the feasible domain, that is, the set of intervals where all possible solutions must reside. On the split complex plane, the aforementioned theories can also be applied similarly.

2.3 Moduli of Split Complex Numbers

The definition of modules over a split complex field is as follows:

$$|\zeta|_{\mathbb{R}^{1,1}} = |uj_+ + vj_-|_{\mathbb{R}^{1,1}} = |u|j_+ + |v|j_- \tag{18}$$

2.4 Convergence Split Complex Sequence

Definition2.4.1 [6] In the split complex plane $\mathbb{R}^{1,1}$,there is a convergent sequence $\{\zeta_n\}_{n\in\mathbb{N}}$of split complex numbers$\mathbb{R}^{1,1}$-converges to the split complex numbersZ_0, if for any strictly positive split complex ε, there exists a natural number$N \in \mathbb{N}$ such that the property holds for all $n \geq N$,we have

$$|\zeta_n - \zeta|_{\mathbb{R}^{1,1}} \prec \varepsilon \tag{19}$$

in which we have

$$\zeta_n = u_{1n} \cdot j_+ + v_{2n} \cdot j_- \tag{20}$$

$$\zeta = u_1 \cdot j_+ + v_2 \cdot j_- \tag{21}$$

$$\varepsilon = \varepsilon_1 \cdot j_+ + \varepsilon_2 \cdot j_- \tag{22}$$

we can equivalently state that

$$|u_{1n} - v_1|_{\mathbb{R}^{1,1}} < \varepsilon_1 \quad \text{and} \quad |u_{2n} - v_2|_{\mathbb{R}^{1,1}} < \varepsilon_2 \tag{23}$$

3. Results

The four operations of separating complex numbers will lay a foundation for the proof of Cauchy convergence criterion and monotone boundedness theorem on split complex field

Theorem 3.1[Four operations on the separation of split complex field] Let$\{\zeta_n\}$and $\{\omega_n\}$be split complex numbers sequences, let$\zeta_n = u_{1n} \cdot j_+ + v_{2n} \cdot j_-and\omega_n = \omega_{1n} \cdot j_+ + \omega_{2n} \cdot j_-$.If

$$\lim_{n\to\infty} \zeta_n = \zeta, \lim_{n\to\infty} \omega_n = \omega \tag{24}$$

in which $\zeta = u \cdot j_+ + v \cdot j_-$, $\omega = \omega_1 \cdot j_+ + \omega_2 \cdot j_-$ and $\varepsilon = (\varepsilon_1 + \varepsilon_3) \cdot j_+ + (\varepsilon_2 + \varepsilon_4) \cdot j_-$ .Our conclusions are as follows:

$$(I) \lim_{n\to\infty} (\zeta_n + \omega_n) = \zeta + \omega,$$

$$(\text{II}) \lim_{n\to\infty} \zeta_n \cdot \omega_n = \zeta \cdot \omega. \tag{25}$$

Proof :

(1)By (24), we have $\forall \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0, \exists N \in N^+, \forall n > N$we have

$$|u_{1n} - u| < \varepsilon_1, |v_{2n} - v| < \varepsilon_2, |\omega_{1n} - \omega_1| < \varepsilon_3, |\omega_{2n} - \omega_2| < \varepsilon_4, \tag{26}$$

so we have

$$\begin{aligned} \left|\zeta_n + \omega_n - \zeta - \omega\right|_{\mathbb{R}^{1,1}} &= \left|u_{1n} \cdot j_+ + v_{2n} \cdot j_- + \omega_{1n} \cdot j_+ + \omega_{2n} \cdot j_- - u \cdot j_+ - v \cdot j_- - \omega_1 \cdot j_+ - \omega_2 \cdot j_-\right|_{\mathbb{R}^{1,1}} \\ &\preceq \left|u_{1n} - u + \omega_{1n} - \omega_1\right| j_+ + \left|v_{2n} - v + \omega_{2n} - \omega_2\right| j_- \\ &= \left|u_{1n} - u\right| j_+ + \left|v_{2n} - v\right| j_- + \left|\omega_{1n} - \omega_1\right| j_+ + \left|\omega_{2n} - \omega_2\right| j_- \\ &\preceq \left(\varepsilon_1 + \varepsilon_3\right) \cdot j_+ + \left(\varepsilon_2 + \varepsilon_4\right) \cdot j_- \\ &= \varepsilon, \end{aligned} \tag{27}$$

According to the arbitrariness of ε, we get (I)

(2) Similar discussion as above, we have

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$$\begin{aligned}\left|\zeta_n\cdot\omega_n-\zeta\cdot\omega\right| &= \left|u_{1n}\omega_{1n}j_+ + v_{2n}\omega_{2n}j_- - u\omega_1 j_+ - v\omega_2 j_-\right| \\ &\preceq \left|u_{1n}\omega_{1n}-u\omega_1\right| j_+ + \left|v_{2n}\omega_{2n}-v\omega_2\right| j_- \\ &= \left|\left(u_{1n}-u\right)\omega_{1n}+u\left(\omega_{1n}-\omega_1\right)\right| j_+ + \left|\left(v_{2n}-v\right)\omega_{2n}-v\left(\omega_{2n}-\omega_2\right)\right| j_- \\ &\preceq \left(\varepsilon_1\left|\omega_{1n}\right|+\varepsilon_2\left|u\right|\right) j_+ + \left(\varepsilon_3\left|\omega_{2n}\right|+\varepsilon_4\left|v\right|\right) j_- \\ &\le \varepsilon M. \end{aligned} \tag{28}$$

Where $M = max\{|u|, |v|, |w_1| + 1, |w_2| + 1\}$. According to the arbitrariness of ε, we have(II).

4. Conclusions

This article elucidates the four arithmetic operations for the separation within the split complex field and derives the convergence split complex sequence. It lays the groundwork for key theorems on the split complex plane, including the squeeze theorem, monotone convergence theorem, Cauchy criterion for convergence, boundedness theorem, and Heine-Borel theorem. Furthermore, it invigorates the application of split complex numbers in the realm of physics by providing a robust mathematical foundation for their theoretical exploration and practical utility.

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