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Generalized Derivations of Leibniz Triple System

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ABSTRACT. *This study introduces definitions of generalized derivation, quasi-derivation and quasi-centroid. Relations among these definitions as well as some basic properties were studied.*

KEYWORDS: *Leibniz triple system, Generalized derivation, Quasiderivation, Centroid, Quasicentroid*

1. Introduction

Bremner and Sanchez-ortega [1] gained the Leibniz triple system by applying the Kolesnikov-Pozhidaev algorithm to the Lie triple system. Cao Yan, Du Desheng and Chen Liangyun [2] proposed the definition of centroid of Leibniz triple system and proved some relevant results. Zhou Jia and Wang Zenghui [3] proved some conclusions of generalized derivation of Leibniz algebra. Since the relation between Leibniz triple system and Leibniz algebra is similar with the relation between the Lie triple system and Lie algebra and the Leibniz triple system is an extension of the Lie system, it is naturally to doubt whether some results of the Lie triple system and Leibniz algebra are true in Leibniz triple system. The definitions of generalized derivation and quasiderivation of the Leibniz triple system have been proposed in reference [4-5].

Definition 1.1[1] A Leibniz triple system T is a vector space over a field F endowed with a trilinear operation $\{\cdot,\cdot,\cdot\}$, satisfying

$$\{a,\{b,c,d\},e\}=\{\{a,b,c\},d,e\}-\{\{a,c,b\},d,e\}-\{\{a,d,b\},c,e\}+\{\{a,d,c\},b,e\} \quad (1)$$

$$\{a,b,\{c,d,e\}\}=\{\{a,b,c\},d,e\}-\{\{a,b,d\},c,e\}-\{\{a,b,e\},c,d\}+\{\{a,b,e\},d,c\} \quad (2)$$

for all $a,b,c,d,e\in Y$.

Note, if T is a Leibniz algebra with product $[\cdot,\cdot]$, then T becomes a Leibniz triple system by putting $\{x,y,z\}=[[x,y],z]$.

Definition 1.2[3] Let $(T,[,])$ be a binary group, where T is a linear space

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over a field K . The multiplication $[,]: T\times T\to T$ meets bilinearity. If $\forall x,y,z\in T$, there's $[x,[y,z]]=[[x,y],z]-[[x,z],y]$. Therefore, $(T,[,])$ is called a Leibniz algebra.

Definition 1.3[2] Let T be a Leibniz triple system over a field F , a derivation of Leibniz triple system T is a linear transformation of $D\in End(T)$ satisfying

$$D(\{x,y,z\})=\{Dx,y,z\}+\{x,Dy,z\}+\{x,y,Dz\},$$

for $\forall x,y,z\in T$.

Let $Der(T)$ be the set of derivation of T . Then, $Der(T)$ is the subalgebra of $End(T)$ and it is called the derivation algebra of T .

Definition 1.4 Let T be a Leibniz triple system over a field F . If $f, f_1, f_2, f_3\in End(T)$, it satisfies

$$\{fx,y,z\}=f_1(\{x,y,z\})-\{x,f_2y,z\}-\{x,y,f_3z\},$$

then, f is called the generalized derivation of T . Let $GDer(T)$ be the set of all generalized derivations of T .

The quasiderivation of T is $D'\in End(T)$ satisfying

$$D'(\{x,y,z\})=\{Dx,y,z\}+\{x,Dy,z\}+\{x,y,Dz\}.$$

Let $GDer(T)$ be the set of generalized derivations and $QDer(T)$ is the set of quasiderivations. Therefore, $GDer(T)$ and $QDer(T)$ are called the generalized derivation algebra and quasiderivation algebra of T .

$$Der(T)\subseteq QDer(T)\subseteq GDer(T)\subseteq End(T).$$

2. Main Results

Theorem 2.1 T is a Leibniz triple system over a field F . Then, the set of quasiderivations $QDer(T)$ is a Lie algebra.

Proof If $f,g\in GDer(T)$, there's $f_1,f_2,f_3,g_1,g_2,g_3\in gl(T)$. Therefore,

$$\{fx,y,z\}=f_1(\{x,y,z\})-\{x,f_2y,z\}-\{x,y,f_3z\},$$

$$\{gx,y,z\}=g_1(\{x,y,z\})-\{x,g_2y,z\}-\{x,y,g_3z\}.$$

Based on above two equations, there's

$$\{fgx,y,z\}=f_1(\{gx,y,z\})-\{gx,f_2y,z\}-\{gx,y,f_3z\}$$

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$$= f_1 g_1(\{x,y,z\}) - f_1(\{x,g_2 y,z\}) - f_1(\{x,y,g_3 z\}) \\ -\{gx, f_2 y, z\} - \{gx, y, f_3 z\}$$

$$= f_1 g_1(\{x,y,z\}) - (\{fx, g_2 y, z\} + \{x, f_2 g_2 y, z\} \\ +\{x, g_2 y, f_3 z\}) - (\{fx, y, g_3 z\} + \{x, f_2 y, g_3 z\} \\ +\{x, y, f_3 g_3 z\}) - \{gx, f_2 y, z\} - \{gx, y, f_3 z\}.$$

Similarly, we can get,

$$\{gfx, y, z\} = g_1 f_1(\{x,y,z\}) - (\{gx, f_2 y, z\} + \{x, g_2 f_2 y, z\} \\ +\{x, f_2 y, g_3 z\}) - (\{gx, y, f_3 z\} + \{x, g_2 y, f_3 z\} \\ +\{x, y, g_3 f_3 z\}) - \{fx, g_2 y, z\} - \{fx, y, g_3 z\}.$$

The above two equations are subtracted and we get,

$$\{(fg - gf)x, y, z\} = (f_1 g_1 - g_1 f_1)\{x,y,z\} - \{x, (f_2 g_2 - g_2 f_2) y, z\} \\ -\{x, y, (f_3 g_3 - g_3 f_3) z\}.$$

Therefore, $fg - gf = [f - g]$ is a generalized derivation. $GDer(T)$ is a Lie algebra.

Definition 1.5 If T is the Leibniz triple system over a field F. Let:

$$C(T) = \{D \in End(T) \mid D(\{x,y,z\}) = \{D(x), y, z\} = \{x, D(y), z\} = \{x, y, D(z)\}, \forall x,y,z \in T$$
.

Then, $C(T)$ is called the centroid of T.

Proposition 2.1 $C(T)$ is the centroid of T. Then, $C(T)$ is the subalgebra of $End(T)$.

Proof Let $D_1, D_2 \in C(T)$. Then, $\forall x,y,z \in T$. We get

$$\{[D_1, D_2](x), y, z\} = \{(D_1 D_2 - D_2 D_1)(x), y, z\} = \{D_1 D_2(x), y, z\} - \{D_2 D_1(x), y, z\}$$

$$= D_1\{D_2(x), y, z\} - D_2\{D_1(x), y, z\} = D_1 D_2\{x,y,z\} - D_2 D_1\{x,y,z\}$$

$$= [D_1, D_2]\{x,y,z\},$$

Similarly, $\{x, [D_1, D_2](y), z\} = [D_1, D_2]\{x,y,z\}$ and $\{x, y, [D_1, D_2]z\} = [D_1, D_2]\{x,y,z\}$.

Hence, $[D_1, D_2] \in C(T)$. Then $C(T)$ is the subalgebra of $End(T)$.

Definition 1.6 Let $(T, \{\cdot,\cdot\})$ be a Leibniz triple system over a field F and $D \in End(T)$. If D satisfies

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$\{D(x), y, z] = \{x, D(y), z\} = \{x, y, D(z)\} = D(\{x, y, z\}) = 0, \forall x, y, z \in T$.

Then, D is called the centroid derivation of T .

Let $ZDer(T)$ be the set of centroid derivations. Then, $ZDer(T)$ is called the centroid derivation algebra of T . It is easy to prove that,

$ZDer(T) \subseteq Der(T) \subseteq QDer(T) \subseteq GDer(T) \subseteq Eed(T)$.

Definition 1.7 Let

$QC(T) = \{D \in End(T) | \{D(x), y, z\} = \{x, D(y), z\} = \{x, y, D(z)\}, \quad \forall x, y, z \in T\}$. Then, $QC(T)$ is a quasicentroid of T .

Obviously, it is easy to prove $C(T) \subseteq QC(T) \subseteq QDer(T)$.

Proposition 2.2 $ZDer(T)$ is an ideal of $Der(T)$.

Proof Let $D_1 \in ZDer(T)$ and $D_2 \in Der(T)$. We can get $\forall x, y, z \in T$,

$$[D_1, D_2](\{x, y, z\}) = D_1 D_2(\{x, y, z\}) - D_2 D_1(\{x, y, z\})$$

$$= D_1(\{D_2 x, y, z\} + \{x, D_2 y, z\} + \{x, y, D_2 z\})$$

$$= 0 ,$$

Since

$$\{[D_1, D_2]x, y, z\} = \{(D_1 D_2 - D_2 D_1)x, y, z\} = \{D_1 D_2 x, y, z\} - \{D_2 D_1 x, y, z\}$$

$$= -(D_2\{D_1 x, y, z\} - \{D_1 x, D_2 y, z\} - \{D_1 x, y, D_2 z\})$$

$$= \{D_1 x, D_2 y, z\} + \{D_1 x, y, D_2 z\} = 0 .$$

Similarly, it can prove $\{x, [D_1, D_2]y, z\} = 0, \{x, y, [D_1, D_2]z\} = 0$.

Therefore, $[D_1, D_2] \in ZDer(T)$. $ZDer(T)$ is an ideal of $Der(T)$.

Proposition 2.3 Let $(T, \{,,\})$ be a Leibniz triple system. Then,

(1) $[Der(T), C(T)] \subseteq C(T)$;

(2) $[QDer(T), QC(T)] \subseteq QC(T)$;

(3) $[QC(T), QC(T)] \subseteq QDer(T)$;

(4) $C(T) \subseteq QDer(T)$;

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(5) $D(Der(T)) \subseteq Der(T), \forall D \in C(T).$

Proof (1) Let $D_1 \in Der(T)$, $D_2 \in C(T)$, $\forall x, y \in T$. It only needs $[D_1, D_2] \in C(T)$ to prove $[Der(T), C(T)] \subseteq C(T)$.

Besides,

$$\{[D_1, D_2]x, y, z\} = D_1 D_2\{x, y, z\} - D_2 D_1\{x, y, z\} = \{D_1 D_2 x, y, z\} - \{D_2 D_1 x, y, z\}$$

Since

$$\{D_1 D_2 x, y, z\} = D_1(\{D_2 x, y, z\}) - \{D_2 x, D_1 y, z\} - \{D_2 x, y, D_1 z\}$$

$$= D_1 D_2(\{x, y, z\}) - D_2(\{x, D_1 y, z\}) - D_2(\{x, y, D_1 z\}),$$

and

$\{D_2 D_1 x, y, z\} = D_2\{D_1 x, y, z\} = D_2 D_1(\{x, y, z\}) - D_2(\{x, D_1 y, z\}) - D_2(\{x, y, D_1 z\})$, so

$\{D_2 D_1 x, y, z\} - \{D_2 D_1 x, y, z\} = D_1 D_2(\{x, y, z\}) - D_2 D_1(\{x, y, z\}) = [D_1, D_2](\{x, y, z\})$.

In other words, $\{[D_1, D_2]x, y, z\} = [D_1, D_2](\{x, y, z\})$.

On the other hand, it has to prove

$\{[D_1, D_2]x, y, z\} = \{x, [D_1, D_2]y, z\} = \{x, y, [D_1, D_2]z\}$. Since

$$\{D_1 D_2 x, y, z\} = D_1(\{x, D_2 y, z\}) - \{D_2 x, D_1 y, z\} - \{D_2 x, y, D_1 z\}$$

$$= \{D_1 x, D_2 y, z\} + \{x, D_1 D_2 y, z\} + \{x, D_2 y, D_1 z\} - \{D_2 x, D_1 y, z\} - \{D_2 x, y, D_1 z\}$$

$$\{D_2 D_1 x, y, z\} = D_2\{D_1 x, y, z\} = \{D_1 x, D_2 y, z\},$$

so,

$$\{D_1 D_2 x, y, z\} - \{D_2 D_1 x, y, z\} = \{x, D_1 D_2 y, z\} - \{x, D_2 D_1 y, z\} = \{x, [D_1, D_2]y, z\},$$

In other words, $\{[D_1, D_2]x, y, z\} = \{x, [D_1, D_2]y, z\}$.

Similarly, it can prove $\{[D_1, D_2]x, y, z\} = \{x, y, [D_1, D_2]z\}$.

Therefore, $[D_1, D_2] \in C(T)$. We can conclude that $[Der(T), C(T)] \subseteq C(T)$.

(2) The poof is similar to the Proof of (1).

(3) Let $D_1, D_2 \in QC(T)$. Then, $\forall x, y, z \in T$. There's

$$\{[D_1, D_2](x), y, z\} + \{x, [D_1, D_2](y), z\} + \{x, y, [D_1, D_2](z)\}$$

$$= \{D_1 D_2(x), y, z\} + \{x, D_1 D_2(y), z\} + \{x, y, D_1 D_2(z)\}$$

$$-\{D_2 D_1(x), y, z\} + \{x, D_2 D_1(y), z\} + \{x, y, D_2 D_1(z)\},$$

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and

$\{D_1D_2(x),y,z\}=\{D_2(x),D_1(y),z\}=\{x,D_2D_1(y),z\}$,

$\{x,D_1D_2(y),z\}=\{x,D_2(y),D_1(z)\}=\{x,y,D_2D_1(z)\}$,

$\{x,y,D_1D_2(z)\}=\{D_1(x),y,D_2(z)\}=\{D_2D_1(x),y,z\}$.

Therefore,

$\{[D_1,D_2](x),y,z\}+\{x,[D_1,D_2](y),z\}+\{x,y,[D_1,D_2](z)\}=0$.

In other words, $[D_1,D_2]\in QDer(T)$.

(4) Let $D\in C(T)$ and $\forall x,y,z\in T$. There's

$D(\{x,y,z\})=\{D(x),y,z\}=\{x,D(y),z\}=\{x,y,D(z)\}$,

Therefore, $\{D(x),y,z\}+\{x,D(y),z\}+\{x,y,D(z)\}=3D(\{x,y,z\})$.

Since

$D'=3D\in End(T)$, $D\in QDer(T)$. In other words, $C(T)\subseteq QDer(T)$.

(5) Let $D_1\in C(T)$ and $D_2\in Der(T)$, $\forall x,y,z\in T$. There's

$D_1D_2\{x,y,z\}=D_1(\{D_2(x),y,z\}+\{x,D_2(y),z\}+\{x,y,D_2(z)\})$

$=\{D_1D_2(x),y,z\}+\{x,D_1D_2(y),z\}+\{x,y,D_1D_2(z)\}$

This means $D_1,D_2\in Der(T)$. Therefore, the conclusion is true.

Theorem 2.2: $(T,\{,,\})$ is a Leibniz triple system and $Z(T)$ is the center of T. Then, $[C(T),QC(T)]\subseteq End(T,Z(T))$. In particular, if $Z(T)=0$, $[C(T),QC(T)]\subseteq\{0\}$.

Proof Let $D_1\in C(T)$ and $D_2\in QC(T)$. Then $\forall x,y,z\in T$ and we can get,

$\{[D_1,D_2](x),y,z\}=\{D_1D_2(x),y,z\}-\{D_2D_1(x),y,z\}$

$=D_1\{D_2(x),y,z\}-\{D_1(x),D_2(y),z\}$

$=D_1\{D_2(x),y,z\}-D_1\{x,D_2(y),z\}$

$=D_1[\{D_2(x),y,z\}-\{x,D_2(y),z\}]=0$,

Since

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$\{y,[D_1,D_2](x),z\}=\{y,D_1D_2(x),z\}-\{y,D_2D_1(x),z\}$

$=D_1\{y,D_2(x),z\}-\{D_2(y),D_1(x),z\}$

$=D_1\{y,D_2(x),z\}-D_1\{D_2(y),x,z\}$

$=D_1[\{y,D_2(x),z\}-\{D_2(y),x,z\}]=0$.

Similarly, it can prove $\{y,z,[D_1,D_2](x)\}=0$. So, $[D_1,D_2](x)\in Z(T)$, and we can get $[D_1,D_2]\in End(T,Z(T))$.

In particular, if $Z(T)=\{0\}$, it is easy to prove $[C(T),QC(T)]=0$.

End

Theorem 2.3 $(T,\{,,\})$ is a Leibniz triple system. If $Z(T)=0$, $QC(T)$ is a Lie algebra if and only if $[QC(T),QC(T)]=0$.

Proof Necessity, if $D_1,D_2\in QC(T)$, and $QC(T)$ is a Lie algebra, then $[D_1,D_2](x)\in QC(T)$. In other words,

$\{[D_1,D_2](x),y,z\}=\{x,[D_1,D_2](y),z\}$,

for $\forall x,y,z\in T$.

It is easy to know from the proof of the proposition 2.3 (3),

$\{[D_1,D_2](x),y,z\}=-\{x,[D_1,D_2](y),z\}$,

Therefore, $\{[D_1,D_2](x),y,z\}=0$. Similarly, we can get $\{x,[D_1,D_2](y),z\}=0$ and $\{x,y,[D_1,D_2](z)\}=0$. Therefore, $[D_1,D_2]=0$.

The sufficiency is proved.

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