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The summery of optimal control methods

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Abstract: *In this essay, we shall present a summary of the methods of optimal control, including variational method, Pontryagin's Maximum Principle, and dynamic programming. First, we will give a brief introduction to optimal control. Then we would like to talk about the general form of optimal control problems. After that, we will discuss three kinds of optimal control methods respectively and point out what types of problems can be solved by them. Finally, we will show the relationship between the three methods.*

Keywords: *summery; optimal; control methods*

1. Introduction

Control theory is application-oriented mathematics that deals with the basic principles un- derlying the analysis and design of (control) systems. Systems can be engineering systems (air conditioner, aircraft, CD player etcetera), economic systems, biological systems and so on[1,2]. To control means that one has to influence the behaviour of the system in a desirable way: for example, in the case of an air conditioner, the aim is to control the temperature of a room and maintain it at a desired level, while in the case of an aircraft, we wish to control its altitude at each point of time so that it follows a desired trajectory[3,4].

These questions of optimality arise naturally. For example, in the case of an aircraft, we are not just interested in flying from one place to another, but we would also like to do so in a way so that the total travel time is minimized or the fuel consumption is minimized. With our algebraic equation $x + u = 10$, in which we want $x < 5$, suppose that furthermore we wish to do so in manner such that u is the least possible integer. Then the only possible choice of the (input) u is 6. Optimal control addresses similar questions with differential equations of the type

$$x'(t) = g(t; x(t); u(t)),$$

together with a performance index functional, which is a function that measures

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optimality.

2. The general form of the basic optimal control problem

There is a system that can be described by the following differential equations:

$$\frac{dx}{d}(t) = g(t;x(t);u(t)), x(t_i) = x_i, t \in [t_i, t_f]$$

in which $x \in R^n andu \in (C[t_i, t_f]^m)$

That is, x is a real vector of n dimensions and each component of u is a continuous function on $[t_i, t_f]$. It is also assumed that g1,...,gn possess partial derivatives with respect to xk, $1 \leq k \leq n$ and ul, $1 \leq l \leq m$ and these are continuous.(So g is continuously differentiable in both variables.)The initial value of x is specified (xi at time ti), which means that specifying u(t) for t $\in [t_i, t_f]$ determines x.

The basic optimal control problem is to choose the control $u \in (C[t_i, t_f])$m such that: The state x is transferred from xi to a state at terminal time tf where some (or all or none)of the state variable components are specified; for example, without loss of generality $x(t_f)k$ is specified for $k \in \{1,...,r\}$.

The functional is minimized $I_{xi}(u) = \int_{ti}^{tf} f(t, x(t), u(t))dt$

3. Calculus of variations

3.1 Introduction

In order to solve this problem, we first make the problem more abstract by considering the problem of finding extremal points $x^* \in X$ for a functional $I : X \to$ R, where X is a normed linear space. We develop a calculus for solving such problems. This situation is entirely analogous to the problem of finding extremal points for a differentiable function f: R → R.

Consider for example the quadratic function $f(x) = ax^2+bx+c$. Suppose that one wants to know the points x^* at which f assumes a maximum or a minimum. We know that if f has a maximum or a minimum at the point x^* , then the derivative of the function must be zero at that point.

We wish to do the same with functionals. In order to do this we need a notion of derivative of a functional, and an analogue of the fact above concerning the necessity of the vanishing

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derivative at extremal points. We define the derivative of a functional I:$X \rightarrow$ R and L is a function of the form of $L(t, x(t), x'(t))$.

$$I(x) = \int_{t_i}^{tf} \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial x'} \delta x' dt$$

We find the derivative of such a functional, and equating it to zero, we obtain a necessary

condition that an extremal curve should satisfy: instead of an algebraic equation, we now obtain a differential equation, called the Euler-Lagrange equation, given by

$$\frac{\partial L}{\partial x}(x^*(t), \frac{dx^*}{dt}(t), t)) - \frac{d}{dt}(\frac{\partial L}{\partial x'}(x^*(t), \frac{dx^*}{dt}(t), t)) = 0, t \in [t_i, t_f]$$

Continuously differentiable solutions $x*$ of this differential equation are then candidates which maximize or minimize the functional I. Historically speaking, such optimization problems arising from physics gave birth to the subject of calculus of variations. The milestone problem, called the "brachistochrone problem", we do not talk about that since it has been discussed in the class. What we want to discuss here is the reason why we define the derivative of functional like this. Actually, it is Euler's creative idea. Now shall we admire the talent of Euler.

3.2 Euler's method

First, consider a functional of the form

$$I(x) = \int_{t_t}^{t_f} L(t, x(t), x(t))dt, x(t_i) = x_i, x(t_f) = x_f$$

Here each curve x is assigned a certain number. To find a related function of the sequence considered in classical analysis, we may proceed as follows. Using the points

$$t_i = t_0, t_1, \cdots, t_n, t_{n+1} = t_f,$$

we divide the interval [ti, tf] into n+1 equal parts. Then we replace the points

$$(t_0, x_i).(t_1, x(t_1)), \cdots, (t_n, x(t_1)), (t_{n+1}, x_f),$$

and we approximate the functional I at x by the sumA

$$In(x_i, \cdots, x_n) = \sum_{k=1}^{n} F(xk, \frac{x_k - x_{k-1}}{})h^{hk} \quad (1)$$

where $x_k = x(t_k)$ and $h_k = t_k - t_{k-1}$. Each polygonal line is uniquely determined by

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the ordinates $x_1, ..., x_n$ of its vertices (recall that $x_0 = x_i$ *and* $x_{n+1} = x_f$ are fixed), and the sum (1) is therefore a function of the n variables $x_1, ..., x_n$. Thus as an approximation, we can regard the variational problem as the problem of finding the extremum of the function $I_n(x_1,...,x_n)$.

In solving variational problems, Euler made extensive use of this method of finite differ- ences. By replacing smooth curves by polygonal lines, he reduced the problem of finding extremum of a functional to the problem of finding extremum of a function of n variables, and then he obtained exact solutions by passing to the limit as $n \to \infty$. In this sense, functionals can be regarded as functions of infinitely many variables (that is, the infinitely many values of x(t) at different points), and the calculus of variations can be regarded as the corresponding analog of differential calculus of functions of n real variables.

Euler was absolutely a genius.

3.3 The fixed points variational problem. Euler-Lagrange equation

The simplest variational problem can be formulated as follows: Let $L(x, x', t)$ be a function with continuous first and second partial derivatives with respect to (x, x', t). Then find $x \in C^1[t_i, t_f]$ such that $x(t_i) = x_i$ *and* $x(t_f) = x_f$, and which is an extremum for the functional

$$I(x) = \int_{t_i}^{t_f} L(t, x(t), x'(t))dt.$$

In other words, the simplest variational problem consists of finding an extremum of a functional, where the class of admissible curves comprises all smooth curves joining two fixed points; see Figure1. We just need to apply the necessary condition for an extremum and solve the Euler-Lagrange equation. Then we will get the optimal solution.

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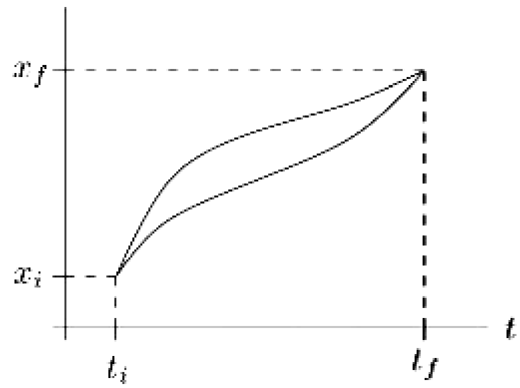

Figure 3.1: Possible paths joining the two fixed points (t_i, x_i) and (t_f , x_f)

3.4 Free boundary conditions

Besides the simplest variational problem considered in the previous section, we now consider the variational problem with free boundary conditions.

Let $L(x, x', t)$ be a function with continuous first and second partial derivatives with respect to(x,x',t). Then find $x \in C^1[t_i, t_f]$ such that $x(t_i) = x_i$ *and* $x(t_f) = x_f$, and which is an extremum for the functional

$$I(x) = \int_{t_i}^{t_f} L(t, x(t), x'(t))dt.$$

The necessary conditions for this problem is that extremum solution x^* should not only satisfy Euler equation but also satisfy the transversality conditions:

$$\frac{\partial L}{\partial x'}\Big|_{t=t_i} = 0 and \frac{\partial L}{\partial x'}\Big|_{t=t_f} = 0$$

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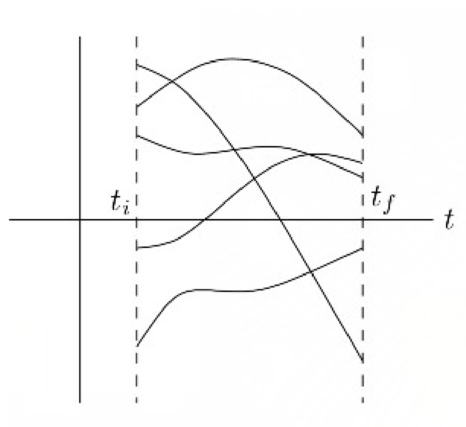

Figure 3.2 Free boundary conditons

4. Pontryagin's Maximum Principle

4.1 The basic optimal control problem

In our basic optimal control problem for ordinary differential equations, we use u(t) for the control and x(t) for the state. The state variable satisfies a differential equation which depends on the control variable:

$$x^{'}(t) = g(t; x(t); u(t))$$

As the control function is changed, the solution to the differential equation will change. Thus, we can view the control-to-state relationship as a map $u(t) \rightarrow \quad x = \quad x(u)$. Our basic optimal control problem consists of finding a piecewise continuous control u(t) and the associated state variable x(t) to maximize the given objective functional, i.e.

$$\max \int_{u_t}^{t_f} f(t, x(t), u(t))dt$$

subject to $x^{'}(t) = g(t; x(t); u(t))\ x(t_i) = x_i$ *and* $x(t_f)$ *free*

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Such a maximizing control is called an optimal control. By $x(t_f)$ free, it is meant that the value of $x(t_f)$ is unrestricted. For our purposes, f and g will always be continuously differentiable functions in all three arguments. Thus, as the control(s) will always be piece- wise continuous, the associated states will always be piecewise differentiable. The principle technique for such an optimal control problem is to solve a set of necessary conditions that an optimal control and corresponding state must satisfy. In practice, one does not need to cope with the necessary conditions for a particular problem, since these conditions can be extended to a version of Pontryagin's Maximum Principle.

4.2 Pontryagin's Maximum Principle

If $u^*(t)$ and $x^*(t)$ are optimal for problem(2), then there exists a piecewise differential adjoint variable $\lambda(t)$ such that

$$H(t,x^*(t),u(t),\lambda(t)) \le H(t,x^*(t),u^*(t),\lambda(t))$$

for all controls u at each time t, where the Hamiltonian H is

$$H = f(t,x(t),u(t)) + \lambda(t)g(t,x(t),u(t))$$

and

$$\frac{\partial H}{\partial U} = 0 \, at \, u^* \qquad (optimality\ equation)$$

$$\lambda'(t) = \frac{\partial H(t,x^*(t),u^*(t),\lambda(t))}{\partial x} \qquad (adjoint\ equation)$$

$$\lambda(t_1) = 0 \ (transversality\ condition)$$

An identical argument generates the same necessary conditions when the problem is minimization rather than maximization. In a minimization problem, we are minimizing the Hamiltonian pointwise, and the inequality in Pontryagin's Maximum Principle in reversed. Indeed, for a minimization problem with f, g being convex in u, we can derive by the same argument as in the case of maximization. We have converted the problem of finding a control that maximizes (or minimizes) the objective functional subject to the

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differential equation and initial condition, to maximizing the Hamiltonian pointwise with respect to the control. Thus to find the necessary conditions, we do not need to calculate the integral in the objective functional, but only use the Hamiltonian. We can also check concavity conditions to distinguish between controls that maximize and those that minimize the objective functional. If

$$\frac{\partial^2 H}{\partial u^2} < 0 \quad at\, u^*,$$

then the problem is maximization, while

$$\frac{\partial^2 H}{\partial u^2} > 0 \quad at\, u^*,$$

goes with minimization. We can view our optimal control problem as having two unknowns, u^* and x^* at the start. We have introduced an adjoint variable λ, which is similar to a Lagrange multiplier. It attaches the differential equation information onto basic optimal control problems the maximization of the objective functional. The following is an outline of how this theory can be applied to solve the simplest problems.

4.3 The procedure of applying the theory

A. Form the Hamiltonian for the problem.

B.Write the adjoint differential equation, transversality boundary condition, and the optimality condition. Now there are three unknowns, u^*, x^* *and* λ.

C. Try to eliminate u^* by using the optimality equation $H_u = 0$, i.e., solve for u^* in terms of x^* *and* λ.

D. Solve the two differential equations for x^* and λ, with two boundary conditions, substituting u^* in the differential equations with the expression for the optimal control from the previous step.

E. After finding the optimal state and adjoint, solve for the optimal control.

5. Dynamic programming

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Bellman and his co-workers pioneered a different approach for solving optimal control prob- lems. Their methods are able to cope with a larger class of control inputs, namely piecewise continuous functions, and they have given sufficient conditions for the existence of an optimal control.

5.1 The optimality principle

The underlying idea of the optimality principle is extremely simple. Roughly speaking, the optimality principle simply says that any part of an optimal trajectory is optimal. We denote the class of piecewise continuous R^m valued functions on $[t_i, t_f]$ by $U[t_i, t_f]$.

(Optimality principle.) Let f(x,u,t) and g(x,u) be continuously differentiable functions of each of their arguments. Let $u \in U[t_i, t_f]$ be an optimal control for the functional

$$I_{xi}(u) = \int_{t_i}^{t_f} f(t, x(t), u(t))dt,$$

subject to

$$x'(t) = g(t, x(t), u(t)), t \in [t_i, t_f], x(t_i) = x_i \tag{3}$$

Let x^* be the corresponding optimal state. If $t_m \in [t_i, t_f]$, then the restriction of u^* to $[t_m, t_f]$ is an optimal control for the functional:

$$I_{x^*}(tm)^{(u)} = \int_{t_m}^{t_f} f(t, x(t), u(t))dt,$$

, subject to

$$x'(t) = g(t, x(t), u(t)), t \in [t_m, t_f], x(t_m) = x^*(t_m) \tag{4}$$

Furthermore,

$$\min_{\substack{u \in U[t_i, t_f] \\ \textit{subject to } (3)}} I_{xi}(u) = \int_{t_m}^{t_f} f(t, x(t), u(t))dt + \min_{\substack{u \in U[t_m, t_f] \\ \textit{subject to } (4)}} I_x{}^*(t_m)^{(u)}$$

5.2 Bellman's equation

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In this section we will give theorem below, which gives a sufficient condition for the existence of an optimal control in terms of the existence of an appropriate solution to Bellmans equation (5).

5.2.1 Let f(x,u,t) and g(x,u,t) be continuously differentiable functions of each of their arguments. Suppose that there exists a function W :$R^n \times [t_i, t_f] \rightarrow R$ such that:

1. W is continuous on $R^n \times [t_i, t_f]$.

2. W is continuously differentiable in$R^n \times [t_i, t_f]$.

3. W satisfies Bellman's equation

$$\frac{\partial w}{\partial t}(x,t)+\min[\frac{\partial w}{u \in R^m \partial t}(x,t)g(x,u,t)+f(x,u,t)]=0,(x,t)\in R^n \times (t_i,t_f) \qquad (5)$$

$$W(x,t_f) = 0 \text{ for all } x \in R^n$$

Then the following implications hold: 1.If $t_m \in [t_i, t_f)$ and $u \in U[t_m, t_f]$, then

$$\int_{t_m}^{t_f} f(t,x(t),u(t))dt \geq w(x_m,t_m),$$

where x is the unique solution to $x'(t) = g(t, x(t), u(t)), x(t_m) = x_m, t \in [t_m, t_f]$

(a)If there exists a function v:$R^n \times [t_i, t_f] \rightarrow R^m$ such that: (a)For all $(x, t) \in R^n \times [t_i, t_f]$

$$\min[\frac{\partial w}{u \in R^m \partial t}(x,t)g(x,u,t)+f(x,u,t)]\frac{\partial w}{\partial t}(x,t)g(x,v(x,t),t)+f(x,v(x,t),t)$$

(b)The equation

has a solution x^*

$x'(t) = g(t, x(t), v(x(t), t)), x(t_i) = x_i, t \in [t_i, t_f]$

(c)u^* defined by $u^*(t) = v(x^*(t), t), t \in [t_i, t_f]$ is a element in $U[t_i, t_f]$. Then u^* is an optimal control for the cost functional I_{xi} defined by

$$I_{xi}(u)=\int_{t_i}^{t_f} f(t,x(t),u(t))dt$$

where x is the unique solution to$x'(t) = g(t, x(t), u(t)), x(t_i) = x_i, t \in [t_i, t_f]$, and further-more,

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$$I_{xi}(u) = \int_{t_i}^{t_f} f(t, x^*(t), u^*(t))dt$$

Let v be the function of part 2. If for each $t_m \in [t_i, t_f)$ and each $x_m \in R^n$, the equation

$$x'(t) = g(t, x(t), v(x(t), t)), x(t_m) = x_m, t \in [t_m, t_f]$$

has a solution, then W is the value function V defined by

$$V(xm, tm) = \min_{u \in u[t_m, t_f]} \int_{t_m}^{t_f} f(t, x(t)),$$

6. The relation of the three methods

Variational method, Pontryagin's Maximum Principle(In this part we call it minimum value principle.) and dynamic programming method are three basic methods for dealing with optimal control problems. The other theoretical methods of optimal control are based on them. These three basic methods are closely related and have difference.

The variational method provides the basic theory and method for the optimal control theory. The variational method can deal with the case where the control constraint is an open set and the Hamilton function exists for the continuous partial derivative of the control. If the control constraint is not an open set, the optimal solution is required to be an inner point. Even if these conditions are satisfied, according to the conclusion of the variational method, there is no guarantee that the optimal solution can be obtained. For example, the Hamilton function is liner function with regard to control.

The principle of minimum value can be considered as a direct generalization of the varia- tional method, which can deal with the case where the control constraint is a closed set and the optimal control Hamilton function does not exist for the continuous partial derivative of the control.

In some cases, the conclusion of variational methods is the generalization of the conclusion of the minimum value principle. Because of the universality of closed set control constraint, the applied range of minimum value principle are much more widespread.

We can deduce conclusions of the dynamic programming method independently, but when the minimum cost function has a quadratic continuous partial derivative for all its variables, all the conclusions of the minimum principle can be easily deduced from that.

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Under certain conditions, the dynamic programming method gives sufficient conditions for the optimal solution, and the optimal control is often the form of state feedback.

The Bellman equation is a partial differential equation. In general, it is difficult to solve. However, for the linear quadratic optimal control problem, we can easily give the optimal solution, and it plays an important role in the demonstration of some related theories. The requirement for the invariability of the optimal cost function is the main limitation of the dynamic programming method.

For the discrete-time system, the inverse recursive solution problem given by the dynamic programming method is superior to the two-point boundary value problem given by the variational method and the minimum value principle.

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