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# The Abelian Group in Mapping Perspective

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***Abstract:*** *The Abelian group, also known as the exchange group, is a class of groups which satisfy the exchange law. This paper reviews Abelian groups from the perspective of mappings and gives sufficient and necessary conditions for a group to be an Abelian group under several special mappings.*

***Keywords:*** *Group, Mapping, Exchange law, Abelian group, Exchange group*

## 1. Introduction

The group is the basic and important type of algebraic system. Let $(S,*)$ be an algebraic system, which is a group if it satisfies the following properties [1].

(1) The law of union, i.e. $\forall a,b,c\in S$, $a*(b*c)=(a*b)*c$;

(2) There exists one unit element, i.e., $\exists e\in S$, $\forall a\in S$ satisfying $e*a=a*e=a$;

(3) There exists an inverse element, i.e., $\forall a\in S$, there exists $a^{-1}\in S$, such that $a*a^{-1}=a^{-1}*a=e$ holds.

Moreover, a group is denoted as an Abelian group, also called exchange group, if $(S,*)$ satisfies the exchange law, i.e., $\forall a,b\in S$, $a*b=b*a$ always holds [2].

A mapping is a correspondence between two sets [3]. Specifically, for a given set $X$ and $Y$, a corresponding rule $f$ is said to be a mapping from set $X$ to set $Y$, denoted as $f:X\to Y$ or $x\to y=f(x)$, if each element $x$ in the set finds a unique element $y$ in $Y$ corresponding to it according to some rule $f$.

Abelian groups are an important object of study in group theory. Earlier studies on the Abelian group can be found in the literature [4]. In 1996, Weidong Gao [5] proved that there exists a nonempty subsequence of any sequence of elements of a finite Abelian group of $n$-order whose sum of elements is zero. In 2006, Kharlampovich Olga et al [6] demonstrated that any two free non-abelian groups have the same fundamental theory and that the theory is decidable. In 2007, Ben Green et al [7] discussed Freiman theorem in arbitrary Abelian groups. In 2010, Marius Tărnăuceanu [8] gave an arithmetic method for computing subgroups of finite Abelian groups. In 2015, Marcin Bownik et al [9] studied the structure of translational invariant spaces on locally tight Abelian groups. In 2018, T. Tamizh Chelvam et al [10] inscribed the power diagrams of certain classes of finite abelian groups and obtained some basic properties of the power diagrams. In 2020, Brian Alspach et al [11] studied the question of whether finite Abelian groups are strongly sequenceable. In 2022, Chinnaraji Annamalai [12] discussed Abelian groups under binomial coefficient addition of combinatorial geometric series. In 2023, Mahmoud Benkhalifa [13] proved that a self-homotopy equivalence group of a rational space cannot be a free Abelian group.

It can be seen that many important results on Abelian groups have been obtained by previous authors. In this paper, the properties and theorems of the existence of Abelian groups are studied under the mapping perspective, with sufficient and necessary conditions given for a group to be an exchange group under some special mappings. In Section 2, the preparatory knowledge for the study is presented. Some main conclusions and their proofs are given in Section 3. Section 4 summarizes the work of this paper and looks forward to the next step of research.

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## 2. Preliminary

**Definition 1** Let $f$ be a map from set $A$ to set $B$. $\forall x, y \in A$ and $x \neq y$, satisfy $f(x) \neq f(y)$, then $f$ is defined as an injective from $A$ to $B$.

**Definition 2** $e$ is said to be a unit element of an algebraic system $(S,*)$ on the operation "$*$" if there exists an element $e \in S$ such that for any $x \in S$, there is $e*x = x*e = x$.

**Definition 3** Let $(S,*)$ be an algebraic system and "$*$" to be a binary operation. $f$ is said to be a self-homomorphism of $(S,*)$ if there exists a function $f$ such that $f: S \to S$ that satisfies $f(x_1 * x_2) = f(x_1) * f(x_2)$.

**Definition 4** Let $(S,*)$ be a group. $a \in S$, the $n$ th power $a^n$ of $a$ can be defined as

$$a^0 = 1 \tag{1}$$

$$a^{j+1} = a^j * a \quad (j \geq 0) \tag{2}$$

$$a^{-j} = \left(a^{-1}\right)^j \quad (j > 0) \tag{3}$$

For powers, the following conclusions can be drawn ($n$ and $m$ are integers).

$$a^n * a^m = a^{n+m} \tag{4}$$

$$\left(a^n\right)^m = a^{n \times m} \tag{5}$$

## 3. Main conclusions and their proofs

Abelian groups are an important object of group theory research and a class of groups that has not been fully resolved. The mapping perspective of the Abelian group exhibits some meaningful properties. The algebraic meaning implied by these properties helps to deepen the researcher's understanding of Abelian groups.

**Theorem 1** Let $(G,*)$ be a group, then the sufficient condition for it to be an Abelian group is

(1) For any injective $g$, $\forall x_1, x_2 \in G$, $g(x_1 * x_2) = g(x_2 * x_1)$;

(2) $g(x) = x^k, k \in \mathbb{Z}$ is a self-homomorphism of $(G,*)$.

**Proof (1)**

Necessity: It follows from the fact that $(G,*)$ is an Abelian group for

$$\forall x_1, x_2 \in G,\ x_1 * x_2 = x_2 * x_1. \tag{6}$$

Hence, for any injective $g$,

$$g(x_1 * x_2) = g(x_2 * x_1) \tag{7}$$

must be satisfied.

Sufficiency: For any injective $g$, if

$$\forall x_1, x_2 \in G,\ g(x_1 * x_2) = g(x_2 * x_1), \tag{8}$$

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then it must follow that

$$x_1 * x_2 = x_2 * x_1, \tag{9}$$

and $(G,*)$ is an Abelian group.

**Proof (2)**

Necessity: $\forall x_1, x_2 \in G$,

$$g(x_1 * x_2) = (x_1 * x_2)^k, \tag{10}$$

$$g(x_1) * g(x_2) = x_1^k * x_2^k = (x_2 * x_1)^k. \tag{11}$$

Because $(G,*)$ is an Abelian group, hence

$$x_1 * x_2 = x_2 * x_1 \text{, i.e. } (x_1 * x_2)^k = (x_2 * x_1)^k. \tag{12}$$

Hence,

$$g(x_1 * x_2) = g(x_1) * g(x_2). \tag{13}$$

It follows that

$$\forall k \in \mathbb{Z},\ g(x) = x^k \tag{14}$$

is a self-homomorphism of $(G,*)$.

Sufficiency: $\forall x_1, x_2 \in G$,

$$g(x_1 * x_2) = (x_1 * x_2)^k, \tag{15}$$

$$g(x_1) * g(x_2) = x_1^k * x_2^k = (x_2 * x_1)^k. \tag{16}$$

$\forall k \in \mathbb{Z}$,

$$g(x) = x^k \tag{17}$$

is a self-homomorphism of $(G,*)$, then

$$g(x_1 * x_2) = g(x_1) * g(x_2) \text{, i.e., } (x_1 * x_2)^k = (x_2 * x_1)^k. \tag{18}$$

Hence,

$$x_1 * x_2 = x_2 * x_1. \tag{19}$$

It follows that $(G,*)$ is an Abelian group.

Theorem 1 (1) reviews the Abelian group from a uniprojective perspective and describes the sufficient conditions for the Abelian group to hold. For more specific mappings $g(x) = x^k,\ k \in \mathbb{Z}$, Theorem 2 describes the sufficient conditions for a group to be an Abelian group under such a mapping. When $k = -1$, for the mapping $g(x) = x^{-1}$, there exists a similar conclusion.

**Corollary 1** Let $(G,*)$ be a group, then the sufficient condition for it to be an Abelian group is that

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$g(x)=x^{-1}$ is a self-isomorphism of $(G,*)$.

**Proof**

Necessity: $\forall x_1,x_2 \in G$, if $x_1 \neq x_2$, then

$$x_1^{-1} \neq x_2^{-1}\text{, i.e. } g(x_1) \neq g(x_2). \tag{20}$$

Hence, $g$ is an injective. Also $\forall x_3 \in G$, there is element $x_3^{-1}$ in $G$, such that

$$g\left(x_3^{-1}\right)=\left(x_3^{-1}\right)^{-1}=x_3. \tag{21}$$

Hence, $g$ is a surjection. Clearly, $g$ is a bijection.

Since $(G,*)$ is an Abelian group, $\forall x_1,x_2 \in G$, it satisfies

$$g(x_1 * x_2)=(x_1 * x_2)^{-1}=x_2^{-1} * x_1^{-1}=x_1^{-1} * x_2^{-1} \tag{22}$$

Also,

$$g(x_1) * g(x_2)=x_1^{-1} * x_2^{-1}. \tag{23}$$

Hence

$$g(x_1 * x_2)=g(x_1) * g(x_2). \tag{24}$$

$g(x)=x^{-1}$ is a self-isomorphism of $(G,*)$ by combining the proof that $g$ is a bijection.

Sufficiency: If $g(x)=x^{-1}$ is a self-isomorphism of $(G,*)$, then $\forall x_1,x_2 \in G$,

$$g(x_1 * x_2)=g(x_1) * g(x_2). \tag{25}$$

While $g(x_1 * x_2)=(x_1 * x_2)^{-1}=x_2^{-1} * x_1^{-1}$,

$$g(x_1) * g(x_2)=x_1^{-1} * x_2^{-1}. \tag{26}$$

Hence, $x_1^{-1} * x_2^{-1}=x_2^{-1} * x_1^{-1}$. It is obvious that the conclusion that $(G,*)$ is an Abelian group can be obtained.

**Theorem 2** Let $(G,*)$ be a group and $x$ be any element of $G$. If $x^k=e$ ($e$ is a unit element and $k$ is even), then $(G,*)$ is an Abelian group.

**Proof** $\forall x \in G$ and an even $k$,

$$x^k=\left(x^n\right)^2=e,\ n \in \mathbb{Z}. \tag{27}$$

That is, $x^n=x^{-n}$, i.e., $x=x^{-1}$. Hence, $\forall x_1,x_2 \in G$, satisfying

$$x_1 * x_2=(x_1 * x_2)^{-1}=x_2^{-1} * x_1^{-1}=x_2 * x_1, \tag{28}$$

which proves that $(G,*)$ is an Abelian group.

The theorem is a result obtained by further specialization ($k$ is even) of the mapping

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$g(x)=x^k, k\in\mathbb{Z}$. The theorem also has a more concise expression: a group is Abelian if the even powers of any element of the group are equal to the unit element of the group.

**Theorem 3** Let $(G,*)$ be a group. There exists a mapping $g(x)=a^{k_1}*x*b^{k_2}$, where $k_1$ and $k_2$ are arbitrary integers, and $a$ and $b$ are any elements of $G$. If $\forall x_1,x_2\in G$, $g(x_1)*g(x_2)=x_1*x_2$, then $(G,*)$ is an Abelian group.

**Proof** $\forall x_1,x_2\in G$,

$$g(x_1)*g(x_2)=a^{k_1}*x_1*b^{k_2}*a^{k_1}*x_2*b^{k_2}. \tag{29}$$

If $g(x_1)*g(x_2)=x_1*x_2$, then it can be introduced that $*$ satisfies the exchange law on $G$ and

$$a^{k_1}*a^{k_1}=b^{k_2}*b^{k_2}=e \tag{30}$$

where $e$ is the unit element.

If $*$ satisfies the exchange law on $G$, it is obvious that $(G,*)$ is an Abelian group. Moreover, by Theorem 2, $\forall a,b\in G$,

$$a^{k_1}*a^{k_1}=b^{k_2}*b^{k_2}=e, \tag{31}$$

then $(G,*)$ is an Abelian group. It can be proved that $(G,*)$ is an Abelian group.

The theorem gives the condition that a group is an Abelian group under a particular mapping. More specifically, for the mapping $g(x)=a*x*a^{-1}$, the following corollary exists:

**Corollary 2** Let $(G,*)$ be a group. There exists a mapping $g(x)=a*x*a^{-1}$, where $a$ is any element of $G$. Then the sufficient conditions for $(G,*)$ to be an Abelian group is $\forall x_1,x_2\in G$, $g(x_1)*g(x_2)=x_1*x_2$.

**Proof**

Necessity:

$$\forall x_1,x_2\in G,\ g(x_1)*g(x_2)=a*x_1*a^{-1}*a*x_2*a^{-1}=a*x_1*x_2*a^{-1}. \tag{32}$$

$$g(x_1)*g(x_2)=a*a^{-1}*x_1*x_2=x_1*x_2 \tag{33}$$

because $(G,*)$ is an Abelian group.

Sufficiency:

$$\forall x_1,x_2\in G,\ g(x_1)*g(x_2)=a*x_1*a^{-1}*a*x_2*a^{-1}=a*x_1*x_2*a^{-1} \tag{34}$$

It can be deduced that $*$ satisfies the exchange law on $G$ because any element of the group and its inverse element are equal to the unit element after doing the operation, and

$$g(x_1)*g(x_2)=x_1*x_2. \tag{35}$$

Hence, $(G,*)$ is an Abelian group.

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## 4. Conclusion

The above theorem and corollary both give the sufficient conditions for a group to be an Abelian group from the perspective of mapping. Theorem 1 (1) gives the sufficient condition for a group to be an Abelian group under the monocular perspective based on the monocular one-to-one property. Theorem 1 (2), Theorem 2 and Theorem 3 all turn to the study of more specific and concrete mappings. Further, Theorem 1 (2) and Theorem 2 are essentially a study of Abelian groups on a mapping $g\left(x\right)=x^k, k\in\mathbb{Z}$.

In the future, the properties of the Abelian group under certain special mapping conditions will continue to be explored. In addition, there are many valuable research directions for the Abelian group. For example, the homomorphism, isomorphism, self-homology, and self-homology of Abelian groups are to be studied in depth. Besides, some well-known theorems in group theory can be used to solve other problems.

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