ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

Bifurcation Analysis of a Class of Discrete Thomas Type System

Weixuan Xiao

School of Science, Tianjin University of Commerce, Tianjin, 300130, China

Abstract: *We often use differential equations to represent continuous dynamical systems and difference equations to represent discrete dynamical systems. In general, discrete dynamical systems have rich dynamic behaviors. Bifurcation problems of differential systems have been extensively studied.The dynamics of a discrete-time Thomas type system is investigated in the closed first quadrant. It is shown that the system undergoes flip bifurcation and Neimark–Sacker bifurcation in the interior by using a center manifold theorem and bifurcation theory. Numerical simulations are presented not only to illustrate our results with the theoretical analysis, but also to exhibit much more interesting dynamical behavior, including orbits of period 2,4,8 and chaotic sets. These results show far richer dynamics of the discrete model compared with the continuous model.*

Keywords: *Thomas type system; Flip bifurcation; Neimark–Sacker bifurcation*

1. Introduction

The Thomas model is a system of two reaction–diffusion equations which was originally proposed in the context of enzyme kinetics. We have known that dynamical systems with simple dynamical behavior in the constant parameter case display very complex behavior including chaos when they are periodically perturbed [1–8].

In this paper, we consider the discrete Thomas type systems

$$\begin{cases} \dfrac{dx}{dt} = a - x - \dfrac{\rho xy}{1+x+kx^2}, \\ \dfrac{dy}{dt} = c(b-y) - \dfrac{\rho xy}{1+x+kx^2}, \end{cases} \tag{1}$$

Applying the forward Euler scheme to system (1), we obtain the discrete-time Thomas type systems as follows:

$$\begin{cases} x \to x + \delta(a - x - \dfrac{\rho xy}{1+x+kx^2}), \\ y \to y + \delta[c(b-y) - \dfrac{\rho xy}{1+x+kx^2}], \end{cases} \tag{2}$$

This paper is organized as follows. In Section 2, we discuss the existence and stability of fixed points for system (2) in the closed first quadrant R_+^2. In Section 3, we show that there exist some values of parameters such that (2) undergoes the flip bifurcation and the Neimark–Sacker bifurcation in the interior of R_+^2. In Section 4, we present numerical simulations, which not only illustrate our results with the theoretical analysis, but also exhibit the complex dynamical behaviors such as orbits with period 2,4,8 and chaotic sets. A brief discussion is given in Section 5.

2. The existence and stability of fixed points

It is clear that the fixed points of (2) satisfy the following equations:

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

$$\begin{cases} x = x + \delta(a - x - \dfrac{\rho xy}{1 + x + kx^2}), \\ y = y + \delta[c(b - y) - \dfrac{\rho xy}{1 + x + kx^2}], \end{cases}$$

Lemma 2.1. For all parameter values, (2) has three fixed points, the boundary fixed point (a, 0), (0, b) and the positive fixed point (x^*, y^*) where (x^*, y^*) satisfy

$$\begin{cases} a - x^* = \dfrac{\rho x^* y^*}{1 + x^* + kx^{*2}}, \\ c(b - y^*) = \dfrac{\rho x^* y^*}{1 + x^* + kx^{*2}} \end{cases}$$

$$x^* = a - c(b - y^*) \tag{3}$$

Now we study the stability of fixed point. Note that the local stability of a fixed point (x, y) is determined by the modules of eigenvalues of the characteristic equation at the fixed point.

The Jacobian matrix J of the map (2) evaluated at any point (x, y) is given by

$$J(x, y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{4}$$

where

$$a_{11} = 1 + \delta[\frac{(kx^2 - 1)\rho y}{1 + x + kx^2} - 1], a_{12} = \delta(\frac{-\rho x}{1 + x + kx^2})$$
$$a_{21} = \delta(\frac{(kx^2 - 1)\rho y}{(1 + x + kx^2)^2}), a_{22} = 1 + \delta(-c - \frac{-\rho x}{1 + x + kx^2})$$

and the characteristic equation of the Jacobian matrix J (x, y) can be written as

$$\lambda^2 + P(x, y)\lambda + Q(x, y) = 0 \tag{5}$$

In order to discuss the stability of the fixed points of (2), we also need the following lemma, which can be easily proved by the relations between roots and coefficients of a quadratic equation.

Lemma 2.2. Let $F(\lambda) = \lambda^2 + P\lambda + Q.$ Suppose that $F(1) > 0.$ λ_1, λ_2 are two roots of $F(\lambda) = 0.$ Then:

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $Q > 1$;

(iv) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) > 0$ and $P \neq 0, 2$;

(v) λ_1 and λ_2 are complex and $|\lambda_1| = 1$ and $|\lambda_2| = 1$ if and only if $P^2 - 4Q < 0$ and $Q = 1$;

Let λ_1 and λ_2 be two roots of (5), which are called eigenvalues of the fixed point (x, y). We recall some definitions of topological types for a fixed point (x, y). A fixed point (x, y) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink is locally asymptotic stable. (x, y) is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source is locally unstable. (x, y) is called a saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). And

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

(x, y) is called non-hyperbolic if either $|\lambda_1|=1$ or $|\lambda_2|=1$.

This paper mainly studies the fixed point (x^*,y^*) of the map (2).

The characteristic equation of the Jacobian matrix J of the map (2) evaluated at the positive fixed point (x^*,y^*) can be written as

$$\lambda^2-[2+\delta(G-1-c-H)]\lambda+1+\delta(G-1-c-H)+\delta^2GH=0 \tag{6}$$

where

$$G=\frac{(kx^2-1)\rho y}{(1+x+kx^2)^2}, H=\frac{\rho x}{1+x+kx^2}.$$

Let

$$F(\lambda)=\lambda^2-[2+\delta(G-1-c-H)]\lambda+1+\delta(G-1-c-H)+\delta^2GH$$

Then

$$F(1)=\delta^2GH,\quad F(-1)=4+2\delta(G-1-c-H)+\delta^2GH.$$

$$p=G-1-c-H(p<0),\quad q=(c-cG+H)(q>0).$$

$$F(\lambda)=\lambda^2-[2+\delta p]\lambda+1+\delta p+\delta^2q.\ F(1)=\delta^2q,\ F(-1)=4+2\delta p+\delta^2.$$

Using Lemma 2.2 we obtain the local dynamics of the fixed point (x^*,y^*).

Proposition 2.1. Let (x^*,y^*) be the positive fixed point of (2);

(i) It is a sink if the following conditions holds:

(i.1) $-2\sqrt{q}\le p<0$ and $0<\delta<-\frac{p}{q}$;

(i.2) $p<-2\sqrt{q}$ and $0<\delta<\frac{-p-\sqrt{p^2-4q}}{q}$;

(ii) It is a source if one of the following conditions holds:

(ii.1) $-2\sqrt{q}\le p<0$ and $\delta>-\frac{p}{q}$;

(ii.2) $p<-2\sqrt{q}$ and $\delta>\frac{-p+\sqrt{p^2-4q}}{q}$;

(iii) It is a saddle if one of the following conditions holds:

$p<-2\sqrt{q}$ and $\frac{-p-\sqrt{p^2-4q}}{q}<\delta<\frac{-p+\sqrt{p^2-4q}}{q}$;

(iv) It is non-hyperbolic if one of the following conditions holds:

(iv.1) $p<-2\sqrt{q}$ and $\delta=\frac{-p\pm\sqrt{p^2-4q}}{q}$ and $\delta\ne\frac{2}{p},\frac{4}{p}$;

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

(iv.2) $-2\sqrt{q} < p < 0$ and $\delta = -\frac{p}{q}$;

From Lemma 2.2 , we can easily see that one of the eigenvalues of the positive fixed point (x^*, y^*) is -1 and the other is neither 1 nor -1 if (iv.1) of Proposition 2.1 holds. When (iv.2) of Proposition 2.1 holds, we can obtain that the eigenvalues of the positive fixed point (x^*, y^*) are a pair of conjugate complex numbers with modulus 1.

Let

$$F_{B1} = \left\{ (a,b,\rho,k,c,\delta) : \delta = \frac{-p-\sqrt{p^2-4q}}{q}, p < -2\sqrt{q}, a,b,\rho,k,c,\delta > 0 \right\}$$

or

$$F_{B2} = \left\{ (a,b,\rho,k,c,\delta) : \delta = \frac{-p+\sqrt{p^2-4q}}{q}, p < -2\sqrt{q}, a,b,\rho,k,c,\delta > 0 \right\}$$

The fixed point (x^*, y^*) can undergo flip bifurcation when parameters vary in a small neighborhood of F_{B1} or F_{B2}.

Let

$$H_B = \left\{ (a,b,\rho,k,c,\delta) : \delta = \frac{p}{q}, -2\sqrt{q} < P < 0, a,b,\rho,k,c,\delta > 0 \right\}$$

The fixed point (x^*, y^*) can undergo Neimark-Sacker bifurcation when parameters vary in a small neighborhood of H_B.

In the following section, we will investigate the flip bifurcation of the positive fixed point (x^*, y^*) if parameters vary in a small neighborhood of F_{B1} (or F_{B2}), and the Neimark-Sacker bifurcation of (x^*, y^*) if parameters vary in a small neighborhood of H_B.

3. Flip bifurcation and Neimark-Sacker bifurcation

On the basis of the analysis in Section 2, we discuss the flip bifurcation and Neimark-Sacker bifurcation of the positive point (x^*, y^*) in this section, We choose parameter δ as a bifurcation parameter for studying the flip bifurcation and Neimark-Sacker bifurcation of (x^*, y^*) by using the center manifold theorem and bifurcation theory of[32-34,35].

We first discuss the flip bifurcation of (2) at (x^*, y^*) when parameters vary in a small neighborhood of F_{B1}. Similar arguments can be applied to the other case F_{B2}.

Taking parameters (a,b,ρ,k,c,δ_1) arbitrarily from F_{B1}, we consider system (2) with (a,b,ρ,k,c,δ_1), which is described by

$$\begin{cases} x \to x + \delta_1(a - x - \dfrac{\rho xy}{1+x+kx^2}) \\ y \to y + \delta_1[c(b-y) - \dfrac{\rho xy}{1+x+kx^2}] \end{cases} \tag{7}$$

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

The map (7) has a unique positive fixed point (x^*, y^*), whose eigenvalues are $\lambda_1 = -1, \lambda_2 = 1 + \delta_1 p$ with $|\lambda_2| \neq 1$ by proposition 2.1.

Since $(a, b, \rho, k, c, \delta_1) \in F_{B1}$, $\delta_1 = \dfrac{-p - \sqrt{p^2 - 4q}}{q}$. Choosing δ^* as a bifurcation parameter, we consider a perturbation of (7) as follows:

$$\begin{cases} x \to x + (\delta_1 + \delta^*)(a - x - \dfrac{\rho x y}{1 + x + k x^2}) \\ y \to y + (\delta_1 + \delta^*)[c(b - y) - \dfrac{\rho x y}{1 + x + k x^2}] \end{cases} \tag{8}$$

where $|\delta^*| \ll 1$, which is a small perturbation parameter.

Let $u = x - x^*, v = y - y^*$. Then we transform the fixed point (x^*, y^*) of map (8) into the origin. We have

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a_{11}u + a_{12}v + a_{13}u^2 + a_{14}uv + b_1 u\delta^* + b_2 v\delta^* + e_1 u^3 + e_2 u^2 v + b_3 u^2 \delta^* + b_4 uv\delta^* + O((|u| + |v| + |\delta^*|)^4) \\ a_{21}u + a_{22}v + a_{23}u^2 + a_{24}uv + c_1 u\delta^* + c_2 v\delta^* + d_1 u^3 + d_2 u^2 v + c_3 u^2 \delta^* + c_4 uv\delta^* + O((|u| + |v| + |\delta^*|)^4) \end{pmatrix} \tag{9}$$

Where

$$\begin{aligned} &a_{11} = 1 + \delta[\frac{(kx^{*2} - 1)\rho y^*}{(1 + x^* + kx^{*2})^2} - 1], a_{12} = -\frac{\delta\rho x^*}{1 + x^* + kx^{*2}}, a_{13} = \frac{\delta\rho y^*(1 + 3kx^* - k^2x^{*3})}{(1 + x^* + kx^{*2})^3}, a_{14} = -\frac{\delta\rho(1 - kx^{*2})}{(1 + x^* + kx^{*2})^2}, \\ &b_1 = \frac{(kx^* - 1)\rho y^*}{(1 + x^* + kx^{*2})^2} - 1, b_2 = -\frac{\rho x^*}{1 + x^* + kx^{*2}}, b_3 = \frac{\rho y^*(1 + 3kx^* - k^2x^{*3})}{(1 + x^* + kx^{*2})^3}, e_1 = \frac{\delta\rho y^*(k - 4kx^* - 1 - 6k^2x^{*2} + k^3x^{*4})}{2(1 + x^* + kx^{*2})^4}, \\ &b_4 = -\frac{\rho(1 - kx^{*2})}{(1 + x^* + kx^{*2})^2}, e_2 = \frac{\delta\rho(1 + 3kx^* - k^2x^{*3})}{2(1 + x^* + kx^{*2})^3}, a_{21} = \frac{\delta\rho y^*(kx^{*2} - 1)}{(1 + x^* + kx^{*2})^2}, a_{22} = 1 + \delta(-c - \frac{\rho x^*}{1 + x^* + kx^{*2}}), \\ &a_{23} = \frac{\delta\rho y^*(1 + 3kx^* - k^2x^{*3})}{(1 + x^* + kx^{*2})^3}, a_{24} = -\frac{\delta\rho(1 - kx^{*2})}{(1 + x^* + kx^{*2})^2}, c_1 = \frac{(kx^* - 1)\rho y^*}{(1 + x^* + kx^{*2})^2}, c_2 = -c - \frac{\rho x^*}{1 + x^* + kx^{*2}}, \\ &d_1 = \frac{\delta\rho y^*(k - 4kx^* - 1 - 6k^2x^{*2} + k^3x^{*4})}{2(1 + x^* + kx^{*2})^4}, d_2 = \frac{\delta\rho(1 + 3kx^* - k^2x^{*3})}{2(1 + x^* + kx^{*2})^3}, c_4 = -\frac{\rho(1 - kx^{*2})}{(1 + x^* + kx^{*2})^2}, c_3 = \frac{\rho y^*(1 + 3kx^* - k^2x^{*3})}{(1 + x^* + kx^{*2})^3} \end{aligned} \tag{10}$$

and $\delta = \delta_1$.

We construct an invertible matrix

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}$$

and use the translation

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} f(x, y, \delta^*) \\ g(x, y, \delta^*) \end{pmatrix} \tag{11}$$

where

$$\begin{aligned} f(x, y, \delta^*) = &\frac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]}{a_{12}(\lambda_2 + 1)}u^2 + \frac{[a_{14}(\lambda_2 - a_{11}) - a_{12}a_{24}]}{a_{12}(\lambda_2 + 1)}uv - \frac{[b_1(\lambda_2 - a_{11}) - a_{12}c_1]}{a_{12}(\lambda_2 + 1)}u\delta^* + \\ &\frac{[b_2(\lambda_2 - a_{11}) - a_{12}c_2]}{a_{12}(\lambda_2 + 1)}v\delta^* + \frac{[e_1(\lambda_2 - a_{11}) - a_{12}d_1]}{a_{12}(\lambda_2 + 1)}u^3 + \frac{[e_2(\lambda_2 - a_{11}) - a_{12}d_2]}{a_{12}(\lambda_2 + 1)}u^2 v + \frac{[b_3(\lambda_2 - a_{11}) - a_{12}c_3]}{a_{12}(\lambda_2 + 1)}u^2\delta^* + \\ &\frac{[b_4(\lambda_2 - a_{11}) - a_{12}c_4]}{a_{12}(\lambda_2 + 1)}uv\delta^* + O((|u| + |v| + |\delta^*|)^4), \end{aligned}$$

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

$$g(x,y,\delta^*)=\frac{[a_{13}(1+a_{11})+a_{12}a_{23}]}{a_{12}(\lambda_2+1)}u^2+\frac{[a_{14}(1+a_{11})+a_{12}a_{24}]}{a_{12}(\lambda_2+1)}uv-\frac{[b_1(1+a_{11})+a_{12}c_1]}{a_{12}(\lambda_2+1)}u\delta^*+\frac{[b_2(1+a_{11})+a_{12}c_2]}{a_{12}(\lambda_2+1)}v\delta^*+\frac{[e_1(1+a_{11})+a_{12}d_1]}{a_{12}(\lambda_2+1)}u^3+\frac{[e_2(1+a_{11})+a_{12}d_2]}{a_{12}(\lambda_2+1)}u^2v+\frac{[b_3(1+a_{11})+a_{12}c_3]}{a_{12}(\lambda_2+1)}u^2\delta^*+\frac{[b_4(1+a_{11})+a_{12}c_4]}{a_{12}(\lambda_2+1)}uv\delta^*+O((|u|+|v|+|\delta^*|)^4),$$

and

$$u=a_{12}(\tilde{x}+\tilde{y}),v=-(1+a_{11})\tilde{x}+(\lambda_2-a_{11})\tilde{y},uv=-a_{12}(1+a_{11})\tilde{x}^2+[a_{12}(\lambda_2-a_{11})-a_{12}(1+a_{11})]\tilde{x}\tilde{y}+a_{12}(\lambda_2-a_{11})\tilde{y}^2,$$
$$u^2=a_{12}^2(\tilde{x}^2+2\tilde{x}\tilde{y}+\tilde{y}^2),v^2=(1+a_{11})^2\tilde{x}^2-2(1+a_{11})(\lambda_2-a_{11})\tilde{x}\tilde{y}+(\lambda_2-a_{11})^2\tilde{y}^2,u^3=a_{12}^3(\tilde{x}^3+3\tilde{x}^2\tilde{y}+3\tilde{x}\tilde{y}^2+\tilde{y}^3),$$
$$u^2v=a_{12}^2[-(1+a_{11})\tilde{x}^3+(\lambda_2-2-3a_{11})\tilde{x}^2\tilde{y}+(2\lambda_2-1-3a_{11})\tilde{x}\tilde{y}^2+(\lambda_2-a_{11}\tilde{y}^3)],$$
$$uv^2=a_{12}[(1+a_{11})^2\tilde{x}^3+(1+a_{11})(1+3a_{11}-2\lambda_2)\tilde{x}^2\tilde{y}+(\lambda_2-a_{11})(\lambda_2-3a_{11}-2)\tilde{x}\tilde{y}^2+(\lambda_2-a_{11})^2\tilde{y}^3].$$

Next, we determine the center manifold $W^c(0,0,0)$ of (11) at the fixed point (0,0) in a small neighborhood of $\delta^*=0$. From the center manifold theorem, we know that there exists a center manifold $W^c(0,0,0)$, which can be approximately represented as follows:

$$W^c(0,0,0)=\{(\tilde{x},\tilde{y},\delta^*)\in R^3:\tilde{y}=a_1\tilde{x}^2+a_2\tilde{x}\delta^*+a_3\delta^{*2}+O((|\tilde{x}|+|\delta^*|)^3)\},$$

where $O((|\tilde{x}|+|\delta^*|)^3)$ is a function with order at least 3 in the variables, and

$$a_1=\frac{a_{12}[a_{13}(1+a_{11})+a_{12}a_{23}]-(1+a_{11})[a_{14}(1+a_{11})+a_{12}a_{24}]}{1-\lambda_2^2},$$
$$a_2=\frac{a_{12}[b_1(1+a_{11})+a_{12}c_1]-(1+a_{11})[b_2(1+a_{11})+a_{12}c_2]}{a_{12}(1-\lambda_2^2)},$$
$$a_3=0.$$

Therefore, we consider the map which is (11) restricted to the center manifold $W^c(0,0,0)$:

$$F_1:\tilde{x}\to-\tilde{x}+h_1\tilde{x}^2+h_2\tilde{x}\delta^*+h_3\tilde{x}^2\delta^*+h_4\tilde{x}\delta^{*2}+h_5\tilde{x}^3+O((|\tilde{x}|+|\delta^*|)^4) \quad (12)$$

Where

$$h_1=\frac{a_{12}[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]-(1+a_{11})[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}]}{(\lambda_2+1)},$$
$$h_2=\frac{-a_{12}[b_1(\lambda_2-a_{11})-a_{12}c_1]-(1+a_{11})[b_2(\lambda_2-a_{11})-a_{12}c_2]}{a_{12}(\lambda_2+1)},$$
$$h_3=\frac{1}{(\lambda_2+1)}\left\{\begin{matrix}a_{12}[b_3(\lambda_2-a_{11})-a_{12}c_3]+2a_2a_{12}-(1+a_{11})[b_4(\lambda_2-a_{11})-a_{12}c_4]+\\a_2[(\lambda_2-a_{11})-(1+a_{11})]-a_1[b_1(\lambda_2-a_{11})-a_{12}c_1]+a_1[b_2(\lambda_2-a_{11})-a_{12}c_2](\lambda_2-a_{11})\end{matrix}\right\},$$
$$h_4=\frac{1}{a_{12}(\lambda_2+1)}\{-a_{12}a_2[b_1(\lambda_2-a_{11})-a_{12}c_1]+a_2[b_2(\lambda_2-a_{11})-a_{12}c_2](\lambda_2-a_{11})\},$$
$$h_5=\frac{1}{(\lambda_2+1)}\left\{\begin{matrix}2a_1a_{12}[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]+a_1[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}][(\lambda_2-a_{11})-\\(1+a_{11})]+a_{12}^2[e_1(\lambda_2-a_{11})-a_{12}d_1]-a_{12}(1+a_{11})[e_2(\lambda_2-a_{11})-a_{12}d_2]\end{matrix}\right\}.$$

In order for map (12) to undergo a flip bifurcation, we require that two discriminatory quantities α_1 and α_2 are not zero,

where

$$\alpha_1=\left(\frac{\partial^2 f}{\partial\tilde{x}\partial\delta^*}+\frac{1}{2}\frac{\partial F_1}{\partial\delta^*}\frac{\partial^2F_1}{\partial\tilde{x}^2}\right)\Big|_{(0,0)}=h_2$$

and

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

$$\alpha_2 = \left(\frac{1}{6} \frac{\partial^3 F_1}{\partial \tilde{x}^3} + \left(\frac{1}{2} \frac{\partial^2 F_1}{\partial \tilde{x}^2} \right)^2 \right) \Big|_{(0,0)} = h_5 + h_1^2.$$

From the above analysis and the theorem of [9-11], we have the following result.

Theorem 3.1. If $\alpha_2 \neq 0$, the map (2) undergoes a flip bifurcation at the fixed point (x^*, y^*), when the parameter δ varies in a small neighborhood of δ_1. Moreover, if $\alpha_2 > 0$ (reps., $\alpha_2 < 0$), then the period-2 orbits that bifurcation from (x^*, y^*) are stable (resp., unstable).

In Section 4 we will give some values of parameters that $\alpha_2 \neq 0$;thus the flip bifurcation occurs as δ varies (see Fig. 1).

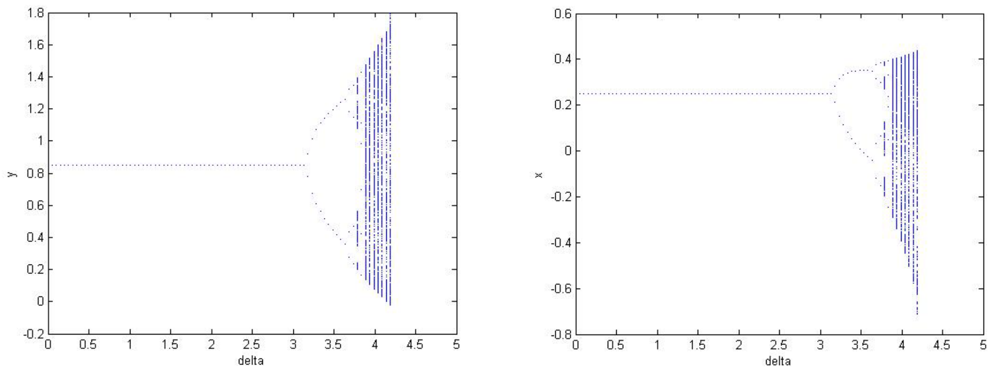

Figure 1: (a)Bifurcation diagram of map(2)in the(δ, x) *plane for*$a = 0.1, b = 0.1, c = 0.2, k = 0.1, rho = 0.5$.*The initial value is*$(0.5,0.6)$.*(b)Bifurcation diagram of map(2)in the* (δ, y) *plane.*

Finally we discuss the Neimark-Sacker bifurcation of (x^*, y^*) if parameters (a,b,ρ,k,c,δ_2) vary in a small neighborhood of H_B. Taking parameters (a,b,ρ,k,c,δ_2) arbitrarily from H_B, we consider system (2) with (a,b,ρ,k,c,δ_2), which is described by

$$\begin{cases} x \to x + \delta_2 (a - x - \dfrac{\rho xy}{1 + x + kx^2}) \\ y \to y + \delta_2 [c(b - y) - \dfrac{\rho xy}{1 + x + kx^2}] \end{cases} \tag{13}$$

The map (13) has a unique positive fixed point (x^*, y^*), where (x^*, y^*) is given in (3).

Since $(a,b,\rho,k,c,\delta_2) \in H_B, \delta_2 = -\dfrac{p}{q}.$ Choosing $\bar{\delta}^*$ as a bifurcation parameter, we consider a perturbation of (13) as follows:

$$\begin{cases} x \to x + (\delta_2 + \bar{\delta}^*)(a - x - \dfrac{\rho xy}{1 + x + kx^2}) \\ y \to y + (\delta_2 + \bar{\delta}^*)[c(b - y) - \dfrac{\rho xy}{1 + x + kx^2}] \end{cases} \tag{14}$$

where $\left| \bar{\delta}^* \right| \ll 1$, which is a small perturbation parameter.

Let $u = x - x^*, v = y - y^*$. Then we transform the fixed point (x^*, y^*) of map (14) into the origin. We have

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a_{11}u + a_{12}v + a_{13}u^2 + a_{14}uv + e_1u^3 + e_2u^2v + O((|u|+|v|)^4) \\ a_{21}u + a_{22}v + a_{23}u^2 + a_{24}uv + d_1u^3 + d_2u^2v + O((|u|+|v|)^4) \end{pmatrix} \tag{15}$$

where $a_{11}, a_{12}, a_{13}, a_{14}, e_1, e_2, a_{21}, a_{22}, a_{23}, a_{24}, d_1, d_2$ are given in (10) by substituting δ for $\delta_2 + \overline{\delta}^*$.

Note that the characteristic equation associated with the linearization of the map (15) at $(u,v) = (0,0)$ is given by

$$\lambda^2 + P(\overline{\delta}^*)\lambda + Q(\overline{\delta}^*) = 0$$

where

$$P(\overline{\delta}^*) = -2 - p(\delta_2 + \overline{\delta}^*),$$
$$Q(\overline{\delta}^*) = 1 + p(\delta_2 + \overline{\delta}^*) + q(\delta_2 + \overline{\delta}^*)^2.$$

Since parameters $(a,b,\rho,k,c,\delta_2) \in H_B$, the eigenvalues of $(0,0)$ are a pair of complex conjugate numbers λ, and $\overline{\lambda}$ with modulus 1 by Proposition 2.1, where

$$\lambda, \overline{\lambda} = -\frac{P(\overline{\delta}^*)}{2} \pm \sqrt{4Q(\overline{\delta}^*) - P^2(\overline{\delta}^*)} = 1 + \frac{p(\delta_2 + \overline{\delta}^*)}{2} \pm \frac{i(\delta_2 + \overline{\delta}^*)}{2}\sqrt{4q - p^2} \tag{16}$$

and we have

$$|\lambda| = \sqrt{Q(\overline{\delta}^*)}, l = \frac{d|\lambda|}{d\overline{\delta}^*}|_{\overline{\delta}^*=0} = -\frac{p}{2} > 0$$

In addition, it is required that when $\overline{\delta}^* = 0, \lambda^m, \overline{\lambda}_m \neq 1 (m = 1,2,3,4)$ which is equivalent to $P(0) \neq -2,0,1,2$. Note that $(a,b,\rho,k,c,\delta_2) \in H_B$. Thus, $P(0) \neq -2,2$. We only need to require that $P(0) \neq 0,1$, which leads to

$$p^2 \neq 2q, 3q \tag{17}$$

Therefore, the eigenvalues $\lambda, \overline{\lambda}$ of fixed point $(0,0)$ of (15) do not lie in the intersection of the unit circle with the coordinate axes when $\overline{\delta}^* = 0$ and (17) holds.

Next, we study the normal form of (15) at $\overline{\delta}^* = 0$.

Let $\overline{\delta}^* = 0, \mu = 1 + \frac{p\delta_2}{2}, \omega = \frac{\delta_2}{2}\sqrt{4q - p^2}$,

$$T = \begin{pmatrix} a_{12} & 0 \\ \mu - a_{11} & -\omega \end{pmatrix},$$

and see the translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

for (15); then the map (15) becomes

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{f}(\tilde{x},\tilde{y}) \\ \tilde{g}(\tilde{x},\tilde{y}) \end{pmatrix} \tag{18}$$

where

$$\tilde{f}(\tilde{x},\tilde{y}) = \frac{a_{13}}{a_{12}}u^2 + \frac{a_{14}}{a_{12}}uv + \frac{e_1}{a_{12}}u^2v + O((|\tilde{x}|+|\tilde{y}|)^4),$$
$$\tilde{g}(\tilde{x},\tilde{y}) = \frac{a_{13}(\mu-a_{11})-a_{12}a_{23}}{a_{12}\omega}u^2 + \frac{a_{14}(\mu-a_{11})-a_{12}a_{24}}{a_{12}\omega}uv + \frac{e_1(\mu-a_{11})-a_{12}d_1}{a_{12}\omega}u^3 + \frac{e_2(\mu-a_{11})-a_{12}d_2}{a_{12}\omega}u^2v + O((|\tilde{x}|+|\tilde{y}|)^4),$$

$$u^2 = a_{12}^2\tilde{x}^2, uv = a_{12}(\mu-a_{11})\tilde{x}^2 - a_{12}\omega\tilde{x}\tilde{y}, u^3 = a_{12}^3\tilde{x}^3, u^2v = a_{12}^2(\mu-a_{11})\tilde{x}^3 - a_{12}^2\omega\tilde{x}^2\tilde{y},$$

and

$$\tilde{f}_{\tilde{x}\tilde{x}} = 2a_{12}a_{13} + 2a_{14}(\mu-a_{11}), \tilde{f}_{\tilde{x}\tilde{y}} = -a_{14}\omega, \tilde{f}_{\tilde{y}\tilde{y}} = 0,$$
$$\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} = 6a_{12}[e_2(\mu-a_{11}) + e_1 + a_{12}], \tilde{f}_{\tilde{x}\tilde{x}\tilde{y}} = -2a_{12}e_2\omega, \tilde{f}_{\tilde{x}\tilde{y}\tilde{y}} = \tilde{f}_{\tilde{y}\tilde{y}\tilde{y}},$$
$$\tilde{g}_{\tilde{x}\tilde{x}} = \frac{2}{\omega}\{a_{12}[a_{13}(\mu-a_{11}) - a_{12}a_{23}] + (\mu-a_{11})[a_{14}(\mu-a_{11}) - a_{12}a_{24}]\},$$
$$\tilde{g}_{\tilde{x}\tilde{y}} = a_{12}a_{24} - a_{14}(\mu-a_{11}), \tilde{g}_{\tilde{y}\tilde{y}} = 0, \tilde{g}_{\tilde{x}\tilde{x}\tilde{x}} = \frac{6a_{12}}{\omega}\{a_{12}[e_1(\mu-a_{11}) - a_{12}d_1] + (\mu-a_{11})[e_2(\mu-a_{11}) - a_{12}d_2]\},$$
$$\tilde{g}_{\tilde{x}\tilde{x}\tilde{y}} = 2a_{12}[a_{12}d_2 - e_2(\mu-a_{11})], \tilde{g}_{\tilde{x}\tilde{y}\tilde{y}} = 0, \tilde{g}_{\tilde{y}\tilde{y}\tilde{y}} = 0.$$

In order for map (18) to undergo Neimark-Sacker bifurcation, we require that the following discriminatory quantity is not zero:

$$a = [-\mathrm{Re}\left(\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\xi_{20}\xi_{10}\right) - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + \mathrm{Re}(\bar{\lambda}\xi_{21})]|_{\bar{\delta}^*=0}, \tag{19}$$

where

$$\xi_{20} = \frac{1}{8}[(\tilde{f}_{\tilde{x}\tilde{x}} - \tilde{f}_{\tilde{y}\tilde{y}} + 2\tilde{g}_{\tilde{x}\tilde{y}}) + i(\tilde{g}_{\tilde{x}\tilde{x}} - \tilde{g}_{\tilde{y}\tilde{y}} - 2\tilde{f}_{\tilde{x}\tilde{y}})], \xi_{11} = \frac{1}{4}[(\tilde{f}_{\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{y}\tilde{y}}) + i(\tilde{g}_{\tilde{x}\tilde{x}} + \tilde{g}_{\tilde{y}\tilde{y}})],$$
$$\xi_{02} = \frac{1}{8}[(\tilde{f}_{\tilde{x}\tilde{x}} - \tilde{f}_{\tilde{y}\tilde{y}} - 2\tilde{g}_{\tilde{x}\tilde{y}}) + i(\tilde{g}_{\tilde{x}\tilde{x}} - \tilde{g}_{\tilde{y}\tilde{y}} + 2\tilde{f}_{\tilde{x}\tilde{y}})], \xi_{21} = \frac{1}{16}[(\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{x}\tilde{y}\tilde{y}} + \tilde{g}_{\tilde{x}\tilde{x}\tilde{y}} + \tilde{g}_{\tilde{y}\tilde{y}\tilde{y}}) + i(\tilde{g}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{g}_{\tilde{x}\tilde{y}\tilde{y}} - \tilde{f}_{\tilde{x}\tilde{x}\tilde{y}} - \tilde{f}_{\tilde{y}\tilde{y}\tilde{y}})].$$

From the above analysis and the theorem in [9-11], we have the following result.

Theorem 3.2. If the condition (17) holds and $a \neq 0$, then map (2) undergoes Neimark-Sacker bifurcation at the fixed point (x^*, y^*) when the parameter δ varies in a small neighborhood of δ_2. Moreover, if $a < 0$ (resp., $a > 0$), then an attracting(resp., repelling) invariant closed curve bifurcation from the fixed point for $\delta > \delta_2$ (resp., $\delta < \delta_2$).

In section 4 we will choose some values of parameters to show the process of Neimark-Sacker bifurcation for map (14) in Fig.3 by numerical simulation.

4. Numerical simulations

In this section, we present the bifurcation diagrams and phase portraits for system (2) to confirm the above theoretical analysis and show the complex dynamical behaviors by using numerical simulations. The bifurcation parameters are considered for the following three cases:

(i) Varying δ in the range $0 < \delta \leq 4.5$, and fixing $a = 0.1, b = 0.1, c = 0.2, k = 0.1, \rho = 0.5$.

(ii) Varying δ in the range $0 < \delta \leq 4.5$, and fixing $a = 0.1, b = 0.1, c = 0.1, k = 0.8, \rho = 0.1$.

For case (i). $a = 0.1, b = 0.1, c = 0.2, k = 0.1, \rho = 0.5$; on the basis of Lemma 2.1, we know that the

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

map (2) has only one positive fixed point. After calculation for the positive fixed point of map (2), the flip bifurcation emerges from the fixed point $(0.25, 0.85)$ a $\delta = 3.1$ with $\alpha_1 = 0.017, \alpha_2 = 0.302$ and $(a,b,c,k,\rho) = (0.1, 0.1, 0.2, 0.1, 0.5) \in F_{B1}$. This shows the correctness of Theorem 3.1.

From Fig. 1(a) and (b), we see that the fixed point is stable for $\delta < 3.1$,and loses its stability at the flip bifurcation parameter value $\delta = 3.1$. We also observe that there is a cascade of period doubling.

The phase portraits which are associated with Fig. 1(a) and (b) are displayed in Fig. 2. For $\delta \in (3.2, 3.76)$, there are orbits of period 2, 4, 8. When $\delta = 4$ we can see the chaotic sets.

For case (ii). $a = 0.1, b = 0.1, c = 0.1, k = 0.8, \rho = 0.1$; according to Lemma 2.1, we know that the map (2) has only one positive fixed point. After calculation of the positive fixed point of map (2), the Neimark–Sacker bifurcation emerges from the fixed point $(0.96, 8.77)$ at $\delta = 3.58$ and its eigenvalues are $\lambda_{\pm} = -1.1176 \pm 1.6337i$. For $\delta = 3.58$, we have $|\lambda_{\pm}| = 1, l = 0.59 > 0, a = -0.8268$ and $(a,b,c,k,\rho) = (0.1, 0.1, 0.1, 0.7, 0.1) \in H_B$. This shows the correctness of Theorem 3.2.

From Fig. 3(a) and (c), we observe that the fixed point $(0.96, 8.77)$ of map (2) is stable for $\delta < 3.58$, that it loses its stability at $\delta = 3.58$, and that an invariant circle appears when the parameter δ less than 3.58. Fig. 3(b) and (d) are local amplifications for $\delta \in [2, 2.5]$.

The phase portraits which are associated with Fig. 3(a) and (c) are displayed in Fig.4, which clearly depicts how a smooth invariant circle bifurcates from the stable fixed point $(0.96, 8.77)$. When δ less than 3.58 there appears a circular curve enclosing the fixed point $(0.96, 8.77)$, and its radius becomes larger with the recession of δ . When δ reaches certain values, for example, $\delta = 2.5$, the circle disappears and a period-14 orbit appears. From Fig.4 we observe that there is attracting chaotic set.

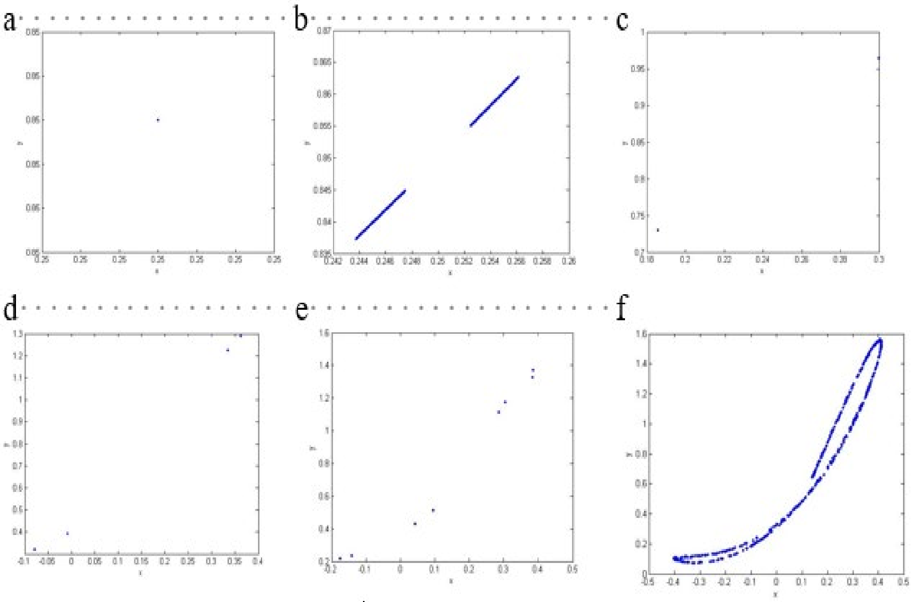

Figure 2: Phase portraits for various of δ corresponding to Fig.1(a). (a)δ=3.1, (b) δ=3.17, (c) δ=3.2, (d)δ=3.65, (e) δ=3.76,(f) δ=4

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

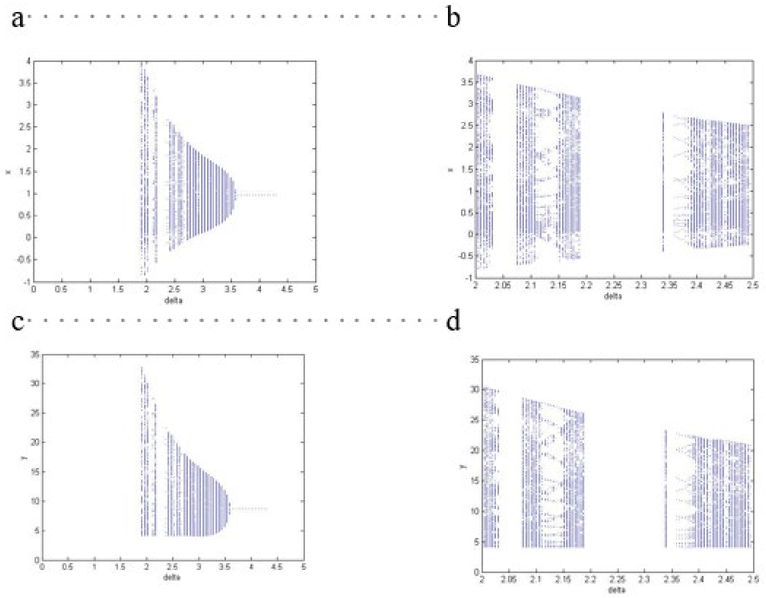

Figure 3: (a) Bifurcation diagram of map (2) in the (δ,x) plane for a=0.1,b=0.1,c=0.1,k=0.7,rho=0.1. The initial value is (0.5,0.7).(b) Local amplification corresponding to (a) for δ ∈[2,2.5].(c) Bifurcation diagram of map (2) in the (δ,y) plane. (d) Local amplification corresponding to (a) for δ ∈[2,2.5].

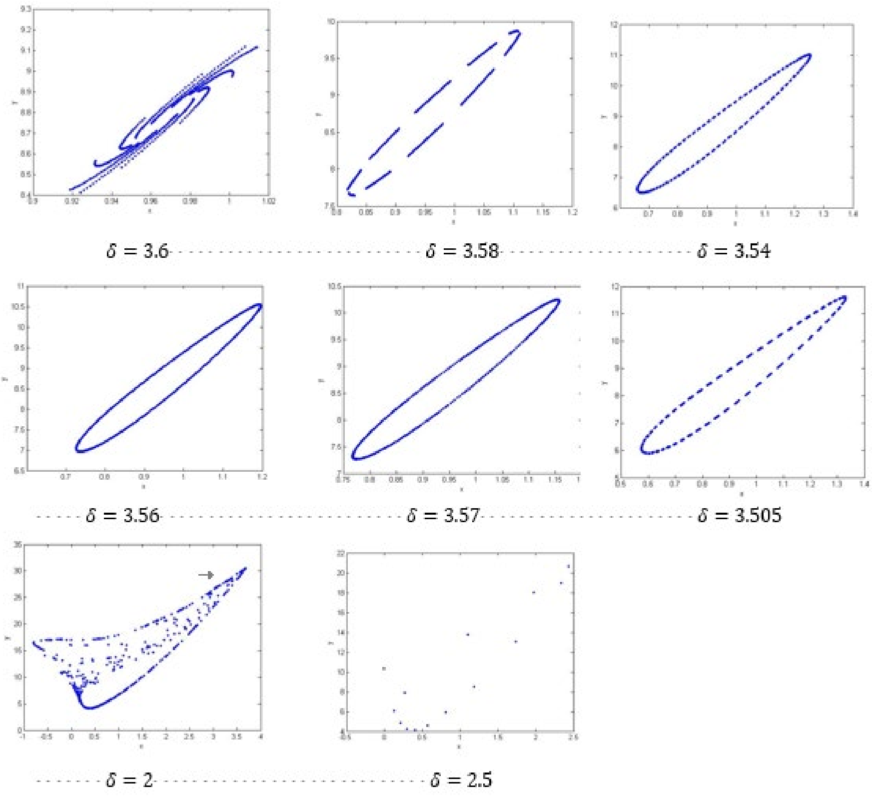

Figure 4: Phase portraits for various values of δ corresponding to Fig.3(a)

5. Conclusion

In this paper, we investigate the complex behaviors of Thomas type systems as a discrete-time dynamical system in the closed first quadrant R_+^2, and show that the unique positive fixed point of (2) can undergo flip bifurcation and Neimark–Sacker bifurcation. Moreover, system (2) displays much more interesting dynamical behavior, including orbits of period 2,4,8 and chaotic sets.

ISSN 2616-5805 Vol. 4, Issue 5: 62-73, DOI: 10.25236/AJMS.2023.040509

References

[1] M. Inoue, H. Kamifukumoto, Scenarios leading to chaos in a forced Lotka–Volterra model, Progr. Theoret. Phys. 71 (1984) 930–937.
[2] Y.A. Kuznetsov, S. Muratori, S. Rinaldi, Bifurcation and chaos in a periodic predator–prey model, Int. J. Bifurcations and Chaos 2 (1992) 117–128.
[3] S. Rinaldi, S. Muratori, Y. Kuznetsov, Multiple attractors, catastrophes and chaos in seasonally perturbed predator–prey communities, Bull. Math. Biol.55 (1993) 15–35.
[4] W.M. Schaffer, B.S. Pederson, B.K. Moore, O. Skarpaas, A.A. King, T.V. Bronnikova, Sub-harmonic resonance and multi-annual oscillations in northern mammals: a non-linear dynamical systems perspective, Chaos Solitons Fractals 12 (2001) 251–264.
[5] G.C.W. Sabin, D. Summers, Chaos in periodically forced predator–prey ecosystem model, Math. Biosci. 113 (1993) 91–113.
[6] D. Summers, J.G. Cranford, B.P. Healey, Chaos in periodically forced discrete-time ecosystem models, Chaos Solitons Fractals 11 (2000) 2331–2342.
[7] J. Vandermeer, Seasonal isochronic forcing of Lotka–Volterra equations, Progr. Theoret. Phys. 96 (1996) 13–28.
[8] J. Vandermeer, L. Stone, B. Blasius, Categories of chaos and fractal boundaries in forced predator–prey models, Chaos Solitons Fractals 12 (2001) 265–276.
[9] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.
[10] C. Robinson, Dynamical Systems: Stability, Symbolic Dynamics and Chaos, 2nd Ed., Boca Raton, London, New York, 1999.
[11] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, New York, 2003.