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Numerical Characteristic Analysis of Order Statistics under Exponential Distribution

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Abstract: *The order statistic is a commonly used statistic, and the order statistic can be used to construct the sample interval, sample mid-range, sample difference and other statistics. This paper systematically analyzes the order statistics under the exponential distribution and the specific forms of the mathematical expectation, variance, covariance and other numerical characteristics of the above statistics constructed by the order statistics, and draws some new conclusions.*

Keywords: *exponential distribution; Order Statistics; sample interval; sample mid-range; numerical feature*

1. Introduction

The exponential distribution is a common life distribution. In life analysis, it is usually necessary to study the numerical characteristics of some incomplete data. Common mathematical expectations, variance, covariance, sample midrange, etc. are some common numerical characteristics. The research results of the above numerical characteristics of exponential distribution are clear, but in sampling, it is often necessary to sort the samples to form order statistics.

Nadarajah S [1] and Thomas P Y [2] studied the computation of moments of the order statistic of beta distribution and the regeneration relation. Kuang N H [3] studied the distribution properties of order statistics of mixed exponential distribution. Balakrishnan and Clifford[4] proposed the application of order statistics and their related inferences in estimation methods. Peihua Jiang [5] studied the density function and the probability distribution of the extreme order statistics in the order of the two-parameter exponential distribution, and discussed its limit distribution. Kuang N H [6] studied the solution of moments of order statistics of two-parameter exponential distribution, and discussed the asymptotic distribution of moments. Wang Ronghua [7] systematically summarized some properties of uniformly distributed population order statistics, including the distribution of order statistics, mathematical expectation and variance, covariance, sample interval, and the distribution of sample mid-range.

Inspired by the research contents of the above scholars, based on the exponential distribution,this paper studies the specific situation of mathematical expectation, variance, covariance, and sample medium range under the order statistics.

2. Preliminaries

Definition 1[8] Let $x_1, x_2, \cdots, x_n$ be a sample taken from the population X , $x_{(i)}$ is called the i order statistic of the sample, Its value is the i observation obtained by arranging the sample observations from small to large. where $x_{(1)} = \min\left(x_1, x_2, \cdots, x_n\right)$ is called the minimum order statistic of the sample, $x_{(n)} = \max\left(x_1, x_2, \cdots, x_n\right)$ is called the maximum order statistic of the sample, and $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ is called the order statistic of the sample.

Definition 2[8]

1) Sample Extreme difference： $R_{n1}^{-} = x_{(n)} - x_{(1)}$.

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2) Sample interval: $R_{ji}^{-}=x_{(j)}-x_{(i)}, i<j$.

3) Sample difference: $R_{i(i-1)}^{-}=x_{(i)}-x_{(i-1)}$.

4) Sample mid-range: $\frac{1}{2}R_{n1}^{+}=\frac{1}{2}\left(x_{(n)}+x_{(1)}\right)$.

Definition 3[8] If the density function of the random variable X is $p(x)=\begin{cases}\lambda e^{-\lambda x}, x\geq 0\\ 0, x<0\end{cases}$, Then X is said to follow a one-parameter exponential distribution. Denoted as $X\sim\exp(\lambda)$, where parameter $\lambda>0$.

The distribution function of the exponential distribution is $F(x)=\begin{cases}1-e^{-\lambda x}, x\geq 0\\ 0, x<0\end{cases}$, The expectation of the exponential distribution is $EX=\frac{1}{\lambda}$, The variance of the exponential distribution is $DX=\frac{1}{\lambda^2}$.

Lemma 1[8] $X\sim\exp(\lambda)$, $x_{(k)}$ is called the k order statistic of the sample, Then the density function of $x_{(k)}$ is $p_k(x)=\frac{n!}{(k-1)!(n-k)!}\left(1-e^{-\lambda x}\right)^{k-1}\left(e^{-\lambda x}\right)^{n-k}\lambda e^{-\lambda x}, x\geq 0$.

Lemma2[9] Where $x_{(1)}$ is called the minimum order statistic of the sample, $x_{(n)}$ is called the maximum order statistic of the sample, The distribution function of $R_{n1}^{-}=x_{(n)}-x_{(1)}$ is $F_{R_{n1}^{-}}(x)=n\int_{-\infty}^{+\infty}\left[F(y+x)-F(y)\right]^{n-1}p(y)dy$, $p_{R_{n1}^{-}}(x)=(n-1)\lambda\left(1-e^{-\lambda x}\right)^{n-2}e^{-\lambda x}, x\geq 0.$

3. Main conclusion

3.1 Solving low-order moments using special constructions

Theorem 1 $X\sim\exp(\lambda), x_1, x_2, \cdots, x_n$ is a sample taken from the population X, $x_{(1)}, x_{(2)}, \cdots, x_{(j)}$ are the first j order statistics, $j=1,2,\cdots,n$,

Order $w_1=nx_{(1)}, w_i=(n-i+1)\left(x_{(i)}-x_{(i-1)}\right), i=2,3,\cdots,j$,then $w_1, w_2, \cdots, w_j$ is an independent and identically distributed random variable, $w_i\sim\exp(\lambda)$.

Proof. From the density function of multiple order statistics, the joint density function of $\left(x_{(1)}, x_{(2)}, \cdots, x_{(j)}\right)$ can be calculated as $\frac{n!}{(n-j)!}\left(\prod_{i=1}^{j}\lambda e^{-\lambda x_{(i)}}\right)\left(e^{-\lambda x_{(j)}}\right)^{n-j}=\frac{n!}{(n-j)!}\lambda^j\exp\left[-\lambda\left((n-j)x_{(j)}+\sum_{i=1}^{j}x_{(i)}\right)\right]$ stem from the title $\sum_{i=1}^{j}w_i=(n-j)x_{(j)}+\sum_{i=1}^{j}x_{(i)}$,using variable transformation $x_{(1)}=\frac{w_1}{n}, x_{(i)}=\frac{w_1}{n}+\frac{w_2}{n-1}+\cdots+\frac{w_i}{n-i+1}, i=2,3,\cdots,j$, Jacobian $|J|=\frac{\partial\left(w_1, w_2, \cdots, w_j\right)}{\partial\left(x_{(1)}, x_{(2)}, \cdots, x_{(j)}\right)}=\frac{n!}{(n-j)!}$,The joint density function of $w_1, w_2, \cdots, w_j$ is $\lambda^j\exp\left[-\lambda\sum_{i=1}^{j}w_i\right], w_i>0, i=1,2,\cdots,j$ .The density function of w_1 is $\lambda\exp\left[-\lambda w_1\right]$,have $w_1\sim\exp(\lambda)$,That is to say, $w_1, w_2, \cdots, w_j$ is an identically distributed random variable, $w_i\sim\exp(\lambda)$.And since the joint density function of $w_1, w_2, \cdots, w_j$ is the product of its marginal

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density functions,Then $w_1, w_2, \cdots, w_j$ is independent of each other.

In summary, $w_1, w_2, \cdots, w_j$ is an independent and identically distributed random variable $w_i \sim \exp(\lambda)$,Proof is complete.

By Theorem 1 we have $Ew_i = \frac{1}{\lambda}, Dw_i = \frac{1}{\lambda^2}$.

Theorem 2 $x_{(j)}$ is the j order statistic in the n samples drawn from $\exp(\lambda)$,then there are $E\left(x_{(j)}\right) = \frac{1}{\lambda}\sum_{i=1}^{j}\frac{1}{n-i+1}, D\left(x_{(j)}\right) = \frac{1}{\lambda^2}\sum_{i=1}^{j}\frac{1}{(n-i+1)^2}$.

Proof. Similar to the construction in Theorem 1. $w_1 = nx_{(1)}, w_i = (n-i+1)\left(x_{(i)} - x_{(i-1)}\right), i = 2,3,\cdots,j$, Using variable transformations to have $x_{(j)} = \frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_j}{n-j+1}$,From the conclusion of Theorem 1, $w_1, w_2, \cdots, w_j$ is an independent and identically distributed random variable, $w_i \sim \exp(\lambda), i = 1,2,\cdots,j$, $Ew_i = \frac{1}{\lambda}, Dw_i = \frac{1}{\lambda^2}$.Using the properties of mathematical expectation and variance, it can be found $E\left(x_{(j)}\right), , D\left(x_{(j)}\right)$.

$$E\left(x_{(j)}\right) = E\left(\frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_j}{n-j+1}\right) = \frac{1}{n}Ew_1 + \frac{1}{n-1}Ew_2 + \cdots + \frac{1}{n-j+1}Ew_j = \frac{1}{\lambda}\sum_{i=1}^{j}\frac{1}{n-i+1}.$$

$$D\left(x_{(j)}\right) = D\left(\frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_j}{n-j+1}\right) = \frac{1}{n^2}Dw_1 + \frac{1}{(n-1)^2}Dw_2 + \cdots + \frac{1}{(n-j+1)^2}Dw_j = \frac{1}{\lambda^2}\sum_{i=1}^{j}\frac{1}{(n-i+1)^2}.$$

Proof is complete.

Special, when $j = 1$, the expectation and variance of the minimum order statistic $x_{(1)}$ can be found $E\left(x_{(1)}\right) = \frac{1}{\lambda}\frac{1}{n}, D\left(x_{(1)}\right) = \frac{1}{\lambda^2}\frac{1}{n^2}$.

When $j = n$, the expectation and variance of the largest order statistic $x_{(n)}$ can be found. $E\left(x_{(n)}\right) = \frac{1}{\lambda}\sum_{i=1}^{n}\frac{1}{i}, D\left(x_{(n)}\right) = \frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{1}{i^2}$.

Theorem 3 $x_{(i)}$ is the i order statistic in the n samples drawn from $\exp(\lambda)$, $x_{(j)}$ is the j order statistic in the n samples drawn from $\exp(\lambda), i < j$,Then we can find the covariance of $x_{(i)}$ and $x_{(j)}$.

$$\operatorname{cov}\left(x_{(i)}, x_{(j)}\right) = \left[\frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{(n-i+1)^2}\right]\frac{1}{\lambda^2}.$$

Proof. Similar to the construction in Theorem 1 $w_1 = nx_{(1)}$, $w_i = (n-i+1)\left(x_{(i)} - x_{(i-1)}\right), w_j = (n-j+1)\left(x_{(j)} - x_{(j-1)}\right)$, Using the variable transformation method, we have $x_{(i)} = \frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_i}{n-i+1}$, $x_{(j)} = \frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_j}{n-j+1}$,

$$\operatorname{cov}\left(x_{(i)}, x_{(j)}\right) = \operatorname{cov}\left(\frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_i}{n-i+1}, \frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_j}{n-j+1}\right).$$

Using the property of covariance to expand the above formula,

$$\operatorname{cov}\left(x_{(i)}, x_{(j)}\right) = \operatorname{cov}\left(\frac{w_1}{n}, \frac{w_1}{n}\right) + \operatorname{cov}\left(\frac{w_1}{n}, \frac{w_2}{n-1}\right) + \cdots + \operatorname{cov}\left(\frac{w_1}{n}, \frac{w_j}{n-j+1}\right)$$

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$$+\operatorname{cov}\left(\frac{w_2}{n-1},\frac{w_1}{n}\right)+\operatorname{cov}\left(\frac{w_2}{n-1},\frac{w_2}{n-1}\right)+\cdots+\operatorname{cov}\left(\frac{w_2}{n-1},\frac{w_j}{n-j+1}\right)$$

$$\cdots+\operatorname{cov}\left(\frac{w_i}{n-i+1},\frac{w_1}{n}\right)+\operatorname{cov}\left(\frac{w_i}{n-i+1},\frac{w_2}{n-1}\right)+\cdots+\operatorname{cov}\left(\frac{w_i}{n-i+1},\frac{w_j}{n-j+1}\right)$$

The expansion has a total of $i\times j$ covariance additions. But $w_1,w_2,\cdots,w_j$ is an independent and identically distributed random variable, so the covariance of the mixture is all 0, $\operatorname{cov}(w_1,w_1)=\operatorname{cov}(w_2,w_2)=\cdots\operatorname{cov}(w_i,w_i)=Dw_i=\frac{1}{\lambda^2}$,

$$\operatorname{cov}\left(x_{(i)},x_{(j)}\right)=\operatorname{cov}\left(\frac{w_1}{n},\frac{w_1}{n}\right)+\operatorname{cov}\left(\frac{w_2}{n-1},\frac{w_2}{n-1}\right)+\cdots+\operatorname{cov}\left(\frac{w_i}{n-i+1},\frac{w_i}{n-i+1}\right)$$

$$=\frac{1}{n^2}\operatorname{cov}(w_1,w_1)+\frac{1}{(n-1)^2}\operatorname{cov}(w_2,w_2)+\cdots+\frac{1}{(n-i+1)^2}\operatorname{cov}(w_i,w_i)$$

$$=\left[\frac{1}{n^2}+\frac{1}{(n-1)^2}+\cdots+\frac{1}{(n-i+1)^2}\right]\frac{1}{\lambda^2}$$, Proof is complete.

Theorem 4 $x_{(i)}$ is the i order statistic in the n samples drawn from $\exp(\lambda)$, $x_{(j)}$ is the j order statistic in the n samples drawn from $\exp(\lambda), i<j$.

Sample interval: $R_{ji}^{-}=x_{(j)}-x_{(i)}, i<j$.

$$ER_{ji}^{-}=E\left(x_{(j)}-x_{(i)}\right)=\left[\frac{1}{n-(i+1)+1}+\frac{1}{n-(i+2)+1}+\cdots+\frac{1}{n-j+1}\right]\frac{1}{\lambda},$$

$$DR_{ji}^{-}=D\left(x_{(j)}-x_{(i)}\right)=\left[\frac{1}{(n-(i+1)+1)^2}+\frac{1}{(n-(i+2)+1)^2}+\cdots+\frac{1}{(n-j+1)^2}\right]\frac{1}{\lambda^2}$$

Proof. Following the construction and notation in Theorem 1, we have $ER_{ji}^{-}=E\left(x_{(j)}-x_{(i)}\right)$

$$=E\left[\left(\frac{w_1}{n}+\frac{w_2}{n-1}+\cdots+\frac{w_j}{n-j+1}\right)-\left(\frac{w_1}{n}+\frac{w_2}{n-1}+\cdots+\frac{w_i}{n-i+1}\right)\right]$$

$$=E\left[\frac{w_2}{n-(i+1)+1}+\frac{w_2}{n-(i+2)+1}+\cdots+\frac{w_j}{n-j+1}\right]$$

$$=\left[\frac{1}{n-(i+1)+1}+\frac{1}{n-(i+2)+1}+\cdots+\frac{1}{n-j+1}\right]\frac{1}{\lambda}$$ .The same can be said

$$DR_{ji}^{-}=D\left(x_{(j)}-x_{(i)}\right)=\left[\frac{1}{(n-(i+1)+1)^2}+\frac{1}{(n-(i+2)+1)^2}+\cdots+\frac{1}{(n-j+1)^2}\right]\frac{1}{\lambda^2}.$$

In particular, there are the expectation and variance of the sample range and sample difference in the same case.

$$ER_{n1}^{-}=E\left(x_{(n)}-x_{(1)}\right)=\left[\frac{1}{n-1}+\frac{1}{n-2}+\cdots+\frac{1}{1}\right]\frac{1}{\lambda},$$

$$DR_{n1}^{-}=D\left(x_{(n)}-x_{(1)}\right)=\left[\frac{1}{(n-1)^2}+\frac{1}{(n-2)^2}+\cdots+\frac{1}{1^2}\right]\frac{1}{\lambda^2},$$

$$ER_{i(i-1)}^{-}=E\left(x_{(i)}-x_{(i-1)}\right)=\frac{1}{n-i+1}\frac{1}{\lambda},$$

$$DR_{i(i-1)}^{-}=D\left(x_{(i)}-x_{(i-1)}\right)=\frac{1}{(n-i+1)^2}\frac{1}{\lambda^2}.$$

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Theorem 5 Sample mid-range : $\frac{1}{2}R_{n1}^{+}=\frac{1}{2}\left(x_{(n)}+x_{(1)}\right)$ .Then $E\left(\frac{1}{2}R_{n1}^{+}\right)=\frac{1}{2\lambda}\left(1+\sum_{i=1}^{n}\frac{1}{i}\right)$, $D\left(\frac{1}{2}R_{n1}^{+}\right)=\frac{1}{4\lambda^2}\left(1+\sum_{i=1}^{n}\frac{1}{i^2}\right)$.

Proof. $E\left(\frac{1}{2}R_{n1}^{+}\right)=E\left[\frac{1}{2}\left(x_{(n)}+x_{(1)}\right)\right]=\frac{1}{2}E\left(x_{(n)}+x_{(1)}\right)=\frac{1}{2}\left(1+\sum_{i=1}^{n}\frac{1}{i}\right)E\left(w_1\right)=\frac{1}{2\lambda}\left(1+\sum_{i=1}^{n}\frac{1}{i}\right)$.

$$D\left(\frac{1}{2}R_{n1}^{+}\right)=D\left[\frac{1}{2}\left(x_{(n)}+x_{(1)}\right)\right]=\frac{1}{4}E\left(x_{(n)}+x_{(1)}\right) \ =\frac{1}{4}E\left(\frac{w_1}{n}+\frac{w_2}{n-1}+\cdots+\frac{w_n}{1}+\frac{w_1}{n}\right)$$

$$=\frac{1}{4}\left(1+\sum_{i=1}^{n}\frac{1}{i^2}\right)D\left(w_1\right)=\frac{1}{4\lambda^2}\left(1+\sum_{i=1}^{n}\frac{1}{i^2}\right).$$

3.2 Using the distribution density function to solve higher-order moments

Expectation and variance are only relatively simple first-order moments and second-order moments, and it is more convenient to use the methods described above to solve them. If they are high-order moments, they can be solved by using the distribution density function.

Theorem 6 $X\sim\exp(\lambda)$, $x_{(k)}$ is called the k order statistic of the sample, then $E\left(x_{(k)}{}^{q}\right)$ $=\frac{n!}{(k-1)!(n-k)!\lambda^q}q!\sum_{i=0}^{k-1}\frac{c_{k-1}^{i}(-1)^i}{(n-k+i+1)^{q+1}}$.

Proof. According to Lemma 1, we have

$$p_k\left(x\right)=\frac{n!}{(k-1)!(n-k)!}\left(1-e^{-\lambda x}\right)^{k-1}\left(e^{-\lambda x}\right)^{n-k}\lambda e^{-\lambda x}, x\geq 0.$$

$$E\left(x_{(k)}{}^{q}\right)=\int_0^{+\infty}x^q p_k\left(x\right)dx=\int_0^{+\infty}x^q\frac{n!}{(k-1)!(n-k)!}\left(1-e^{-\lambda x}\right)^{k-1}\left(e^{-\lambda x}\right)^{n-k}\lambda e^{-\lambda x}dx$$

$$=\frac{n!}{(k-1)!(n-k)!}\int_0^{+\infty}\lambda x^q\sum_{i=0}^{k-1}c_{k-1}^{i}(-1)^i\left(e^{-\lambda x}\right)^{n-k+i-1}dx$$

$$\underline{\underline{y=\lambda x}}\frac{n!}{(k-1)!(n-k)!\lambda^q}\int_0^{+\infty}y^q\sum_{i=0}^{k-1}c_{k-1}^{i}(-1)^i\left(e^{-y}\right)^{n-k+i-1}dy$$

$$\underline{\underline{z=(n-k+i+1)y}}\frac{n!}{(k-1)!(n-k)!\lambda^q}\sum_{i=0}^{k-1}\frac{c_{k-1}^{i}(-1)^i}{(n-k+i+1)^{q+1}}\int_0^{+\infty}z^q\left(e^{-z}\right)dz$$

$$=\frac{n!}{(k-1)!(n-k)!\lambda^q}q!\sum_{i=0}^{k-1}\frac{c_{k-1}^{i}(-1)^i}{(n-k+i+1)^{q+1}}.$$

When $k=1,q=1$, $E\left(x_{(1)}\right)=\frac{n!}{(1-1)!(n-1)!\lambda^1}1!\frac{1}{n^2}=\frac{1}{\lambda}\frac{1}{n}$,Consistent with the previous method.

When $k=n,q=1$, $E\left(x_{(n)}\right)=\frac{n!}{(n-1)!(n-n)!\lambda^1}1!\sum_{i=0}^{n-1}\frac{c_{n-1}^{i}(-1)^i}{(n-n+i+1)^2}=\frac{n}{\lambda}\sum_{i=0}^{n-1}\frac{c_{n-1}^{i}(-1)^i}{(i+1)^2}$, It is quite different from the $E\left(x_{(n)}\right)=\frac{1}{\lambda}\sum_{i=1}^{n}\frac{1}{i}$ form obtained by the previous method, but its value is actually the same.

It can also be seen from this that the lower-order rectangular formula solved by the method described above is simpler. The variance can also be calculated using the formula obtained by the distribution density function method here.

$$D\left(x_{(k)}\right)=E\left(x_{(k)}{}^{2}\right)-E\left(x_{(k)}\right)^2,$$

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$$E\left(x_{(k)}{}^2\right)=2\frac{n!}{(k-1)!(n-k)!\lambda^2}\sum_{i=0}^{k-1}\frac{c_{k-1}^i(-1)^i}{(n-k+i+1)^3},$$

$$E\left(x_{(k)}\right)=\frac{n!}{(k-1)!(n-k)!\lambda}\sum_{i=0}^{k-1}\frac{c_{k-1}^i(-1)^i}{(n-k+i+1)^2},$$

When $k=1$, $D\left(x_{(1)}\right)=E\left(x_{(1)}{}^2\right)-E\left(x_{(1)}\right)^2=2\frac{n}{\lambda^2}\frac{1}{n^3}-\left[\frac{n}{\lambda}\frac{1}{n^2}\right]^2=\frac{1}{\lambda^2}\frac{1}{n^2}$, When $k=n$,

$$D\left(x_{(n)}\right)=E\left(x_{(n)}{}^2\right)-E\left(x_{(n)}\right)^2=2\frac{n}{\lambda^2}\sum_{i=0}^{n-1}\frac{c_{n-1}^i(-1)^i}{(i+1)^3}-\left[\frac{n}{\lambda}\sum_{i=0}^{n-1}\frac{c_{n-1}^i(-1)^i}{(i+1)^2}\right]^2$$

$$=\left[\frac{1}{n-1}+\frac{1}{n-2}+\cdots+\frac{1}{1}\right]\frac{1}{\lambda}.$$

Theorem 7 $X\sim\exp(\lambda)$, $E\left(R_{n1}^-\right)^q=\frac{1}{\lambda^q}(n-1)q!\sum_{i=0}^{n-2}(-1)^i c_{n-2}^i\frac{1}{(i+1)^{q+1}}$.

Proof. According to Lemma 2, we have

$$F_{R_{n1}^-}(x)=\left(1-e^{-\lambda x}\right)^{n-1},\ p_{R_{n1}^-}(x)=(n-1)\lambda\left(1-e^{-\lambda x}\right)^{n-2}e^{-\lambda x},x\geq 0.$$

$$E\left(\left(R_{n1}^-\right)^q\right)=\int_0^{+\infty}x^q p_{R_{n1}^-}(x)dx=\int_0^{+\infty}x^q(n-1)\lambda\left(1-e^{-\lambda x}\right)^{n-2}e^{-\lambda x}dx$$

$$=\lambda(n-1)\int_0^{+\infty}x^q\sum_{i=0}^{n-2}(-1)^i c_{n-2}^i e^{(i+1)(-\lambda x)}dx$$

$$\underline{\underline{y=\lambda x(i+1)}}\lambda(n-1)\sum_{i=0}^{n-2}(-1)^i c_{n-2}^i\frac{1}{\left[\lambda(i+1)\right]^{q+1}}\int_0^{+\infty}y^q e^{-y}dy$$

$$\underline{\underline{y=\lambda x(i+1)}}\frac{1}{\lambda^q}(n-1)q!\sum_{i=0}^{n-2}(-1)^i c_{n-2}^i\frac{1}{(i+1)^{q+1}}.$$

when $q=1$, $E\left(R_{n1}^-\right)=\frac{1}{\lambda}(n-1)\sum_{i=0}^{n-2}\frac{(-1)^i c_{n-2}^i}{(i+1)^2}$.

4. Conclusions

This paper studies the numerical characteristics of order statistics and statistics constructed from order statistics under exponential distribution. The main conclusions are $X\sim\exp(\lambda)$, $x_1,x_2,\cdots,x_n$ is a sample taken from the population X, $x_{(1)},x_{(2)},\cdots,x_{(j)}$, are the first j order statistics, $j=1,2,\cdots,n$, Order $w_1=nx_{(1)},w_i=(n-i+1)\left(x_{(i)}-x_{(i-1)}\right),i=2,3,\cdots,j,w_i\sim\exp(\lambda)$.

The expectation and variance of $x_{(j)}$, $E\left(x_{(j)}\right)=\frac{1}{\lambda}\sum_{i=1}^{j}\frac{1}{n-i+1}$, $D\left(x_{(j)}\right)=\frac{1}{\lambda^2}\sum_{i=1}^{j}\frac{1}{(n-i+1)^2}$.

Covariance of $x_{(i)}$ And $x_{(j)}$, $\operatorname{cov}\left(x_{(i)},x_{(j)}\right)=$ $=\left[\frac{1}{n^2}+\frac{1}{(n-1)^2}+\cdots+\frac{1}{(n-i+1)^2}\right]\frac{1}{\lambda^2}$ .sample interval $R_{ji}^-=x_{(j)}-x_{(i)}$, $i<j$,

$$ER_{ji}^-=E\left(x_{(j)}-x_{(i)}\right)=\left[\frac{1}{n-(i+1)+1}+\frac{1}{n-(i+2)+1}+\cdots+\frac{1}{n-j+1}\right]\frac{1}{\lambda},$$

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$DR_{ji}^{-}=D\left(x_{(j)}-x_{(i)}\right)=\left[\frac{1}{\left(n-(i+1)+1\right)^2}+\frac{1}{\left(n-(i+2)+1\right)^2}+\cdots+\frac{1}{\left(n-j+1\right)^2}\right]\frac{1}{\lambda^2}$. Sample mid-range $\frac{1}{2}R_{n1}^{+}=\frac{1}{2}\left(x_{(n)}+x_{(1)}\right)$, $E\left(\frac{1}{2}R_{n1}^{+}\right)=\frac{1}{2\lambda}\left(1+\sum_{i=1}^{n}\frac{1}{i}\right)$, $D\left(\frac{1}{2}R_{n1}^{+}\right)=\frac{1}{4\lambda^2}\left(1+\sum_{i=1}^{n}\frac{1}{i^2}\right)$.The expectation of higher order moments, $E\left({x_{(k)}}^{q}\right)=\frac{n!}{(k-1)!(n-k)!\lambda^{q}}q!\sum_{i=0}^{k-1}\frac{c_{k-1}^{i}\left(-1\right)^{i}}{\left(n-k+i+1\right)^{q+1}}$. These conclusions are of great significance to the study of numerical characteristics of order statistics under exponential distribution. It is instructive to continue thinking about the numerical characteristics of order statistics under other distributions.

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