ISSN 2522-6398 Vol. 8, Issue 4: 60-65, DOI: 10.25236/FER.2025.080409

Exploring determinants expansions through functions

Mingming Zhou[1,a,*], Qiong Wang[1,b], Hengzhe Li[1,c], Yingbin Ma[1,d]

[1]College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan, 453007, China
[a]zhoumming1218@163.com, [b]wang1472581113@qq.com, [c]313507570@qq.com, [d]2745915275@qq.com
**Corresponding author*

Abstract: *The determinant expansion theory is one of the core theories in advanced algebra. It not only provides a crucial basis for determining invertible matrices but also serves as the foundation for important theorems such as Cramer's rule. Compared to the knowledge learned in high school, the determinant expansion theorem is significantly more abstract and logically profound, often appearing esoteric and challenging for students who have just entered university. This article will delve into the intrinsic logic of determinant expansion from the perspective of functions, attempting to reinterpret determinants through a functional lens, thereby offering students a more intuitive approach to understanding.*

Keywords: *Determinant expansion; Algebraic cofactor; Educational reform; Order permutation*

1. Introduction

The study of determinant expansion mechanisms has remained a focal point of research in linear algebra, spanning both theoretical innovation and pedagogical application. Historically, the logical framework for determinant expansion originated from the need to solve systems of linear equations, with its mathematical formalism progressively refined through foundational contributions by Leibniz, Cramer, and Laplace. In traditional pedagogical systems, determinant expansion is typically centered on minors and algebraic cofactors, achieved through recursive expansion or block matrix decomposition for dimensionality reduction.

In recent years, academic exploration of determinant expansion has diversified across multiple dimensions:

Theoretical Advancements: Researchers have investigated expansion mechanisms by integrating summation-by-classification perspectives with permutation parity theory[5]. Conceptual Innovation: Literature[6] introduces a novel conceptualization of determinants and provides alternative proofs for determinant properties and Cramer’s Rule. Geometric Visualization: Studies[7] [8]leverage geometric interpretations in vector spaces (parallelepiped volume representations) to enhance students’ understanding of determinant sign conventions and expansion patterns. This multidimensional evolution not only enriches theoretical frameworks but also bridges abstract algebraic concepts with intuitive geometric insights, fostering deeper comprehension in both academic research and classroom instruction.

As a core foundational topic in linear algebra, determinant expansion and the comprehension of its underlying operational logic often pose significant cognitive challenges for beginners. Given that high school students have systematically mastered the conceptual framework and analytical methods of functions, this paper proposes a function-oriented analytical framework for determinants:

- Deconstructing determinants as multilinear functions with specialized properties.

- Systematically elucidating expansion mechanisms through the fundamental characteristics of this mathematical object and the operational rules of elementary row (column) transformations.

This cross-disciplinary cognitive transfer not only bridges students’ existing knowledge of functions but also reveals the intrinsic principles of determinant expansions through deductive explorations of functional properties. The framework offers novel pedagogical pathways and methodological guidance for teaching determinants, enhancing both conceptual clarity and computational fluency.

ISSN 2522-6398 Vol. 8, Issue 4: 60-65, DOI: 10.25236/FER.2025.080409

2. Explore determinant expansions as functions

We will now commence our investigation. Prior to this, however, it is necessary to establish some foundational definitions.

2.1. The definition of a determinant

Definition 1[1]. n th-order determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

It is equal to the algebraic sum of the products $a_{1j_1}a_{2j_2}\cdots a_{nj_n}$, where each product consists of n elements selected from distinct rows and columns. Here, $j_1 j_2 \cdots j_n$ represents a permutation of $1,2\cdots n$. The term $a_{1j_1}a_{2j_2}\cdots a_{nj_n}$ takes a positive sign if the permutation $j_1 j_2 \cdots j_n$ is even, and a negative sign if the permutation is odd. This definition can be formally expressed as:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1j_1}a_{2j_2}\cdots a_{nj_n} \tag{1}$$

By the definition of a determinant, each term in the determinant contains exactly one element from the i-th row. Thus, the determinant can be partitioned into n groups based on the elements of the i-th row. The remaining part after factoring out a_{ij} from the terms containing a_{ij} is called the cofactor A_{ij} of a_{ij}.

Row (Column) Expansion Theorem for Determinants:

A determinant equals the sum of the products of each element in any row (column) and its corresponding cofactor. Specifically:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \quad (i = 1,2,\ldots,n.) \tag{2}$$

or $$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \quad (j = 1,2,\ldots,n.)$$

For clarity in subsequent explanations, we assume $D = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$.

Definition 2[3]. In an n-th order determinant, the minor M_{ij} of the element a_{ij} is the $n-1$-th order determinant formed by the remaining $(n-1)^2$ elements after removing the i-th row and j-th column, arranged in their original order.

2.2. How to Represent Determinants as Functions

In this section, we will present the main findings discussed in our paper.

The biggest obstacle for students learning the expansion formula of determinants lies in their understanding of the determinant D. They struggle to view D as a function or, psychologically, refuse to accept that D is a function. In fact, according to the definition of the determinant, if we regard $a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, \cdots, a_{nn}$ as independent variables, then D is an n^2-variable function, known as the

ISSN 2522-6398 Vol. 8, Issue 4: 60-65, DOI: 10.25236/FER.2025.080409

determinant function, denoted as $f = D\left(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, \cdots, a_{nn}\right)$. From equation (1), we obtain

$$f = D\left(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, \cdots, a_{nn}\right) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{3}$$

According to equation (2), we have that:

$$f = D\left(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, \cdots, a_{nn}\right) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \tag{4}$$

Before exploring the value of A_{ij}, let us first consider the relationship between the value of A_{11} and the independent variable of function f. In the function expression (4), if we set $a_{11} = 1$ and $a_{12} = a_{13} = \cdots = a_{1n} = 0$, then we obtain $f = A_{11}$, which means that A_{11} equals the function value of f when $a_{11} = 1$ and $a_{12} = a_{13} = \cdots = a_{1n} = 0$. When $a_{11} = 1$ and $a_{12} = a_{13} = \cdots = a_{1n} = 0$, Equation (3) transforms into

$$f = D = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = A_{11} = M_{11}$$

Based on the calculation result of A_{11}, we now turn to consider the value of A_{i1}. If $f = a_{i1}A_{i1}$, then we can derive that $a_{i1} = 1$ and $a_{12} = a_{13} = \cdots = a_{1n} = 0$. We have that

$$f = D = A_{i1} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n} \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,n} \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n} \\ 1 & 0 & 0 & \cdots & 0 \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

From the properties of determinants[4], we know that swapping two adjacent rows or columns of a determinant only changes its sign. Therefore, we can derive the following.

$$f = A_{i1} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n} \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,n} \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n} \\ 1 & 0 & 0 & \cdots & 0 \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

ISSN 2522-6398 Vol. 8, Issue 4: 60-65, DOI: 10.25236/FER.2025.080409

$$=\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{11} & a_{12} & a_{13} & & a_{1n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i-1} M_{i1}$$

Therefore, we can establish the relationship between the independent variables of the function f and A_{i1}, which is $A_{i1} = (-1)^{i-1} M_{i1}$.

Now, let's consider the relationship between the value of A_{ij} and the coefficients of the function f, where $i, j \in \{1, 2, \cdots, n\}$. If $f = A_{ij}$, then similarly to the above, we can derive that $a_{ij} = 1$, and $a_{i1} = a_{i2} = \cdots = a_{i,j-1} = a_{i,j+1} = \cdots a_{in} = 0$. That is,

$$f = A_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

According to the properties of determinants, when two adjacent rows or columns in a determinant are swapped, the sign of the determinant changes, i.e., the positive or negative sign is reversed. Based on this property, we can rearrange the determinant into the following form.

$$f = A_{ij} = (-1)^{i-1} \begin{vmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ a_{1,1} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

ISSN 2522-6398 Vol. 8, Issue 4: 60-65, DOI: 10.25236/FER.2025.080409

$$=(-1)^{i-1+j-1}\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{1,j} & a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,j} & a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,j} & a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{nj} & a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$=(-1)^{i+j}M_{ij}$$

Therefore, through the above steps, we can gradually establish the relationship between the coefficients of the function f and the value of A_{ij}. According to the definition of the determinant, we know the value of each element of the determinant, which allows us to determine the value of A_{ij}. By substituting these values into our function expression

$$f=f(a_{11},a_{12},\ldots,a_{1n},a_{21}\ldots,a_{nn})=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}$$

we can see that in the determinant, knowing the values of $a_{i1},a_{i2},\ldots,a_{in}$ ultimately allows us to determine the magnitude of the function f, which is the value of the determinant.

2.3. Practical Applications.

In the previous section, we have already mastered the methods of representing determinants using functions. Next, in this section, we will delve into concrete examples to demonstrate how to more efficiently and ingeniously apply these functions to compute determinants. Furthermore, we will elaborate on the positive impact of this process on teaching and research activities, thereby promoting the development and innovation of mathematical education. Let me now provide a concrete example to illustrate how to apply functions to compute determinants.

Example 1: Compute the determinant $\begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix}$.

We know the value of this determinant is $x_1x_4-x_2x_3$. Now, let us attempt to interpret this determinant using the concept of functions. We define a four-variable function:

$$f(x_1,x_2,x_3,x_4)=x_1A_{11}+x_2A_{12}+x_3\times 0+x_4\times 0=x_1\times 0+x_2\times 0+x_3A_{21}+x_4A_{22}$$

where x_1,x_2,x_3,x_4 are variables, and $A_{11},A_{12},A_{21},A_{22}$ are constants. Substituting $A_{11}=x_4$, $A_{12}=-x_3$, $A_{21}=-x_3$ and $A_{22}=x_1$, we obtain:

$$f(x_1,x_2,x_3,x_4)=\begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix}=x_1x_4-x_2x_3$$

Example 2: Compute the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

Based on the definition of the above function, we can derive that

$$f(x_{11},x_{12},x_{13},x_{21},\cdots,x_{33})=\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}=x_{i1}A_{i1}+x_{i2}A_{i2}+x_{i3}A_{i3}$$

Here, x_{i1},x_{i2},x_{i3} are independent variables, and A_{i1},A_{i2},A_{i3} are coefficients $i=1,2,3$. During the

ISSN 2522-6398 Vol. 8, Issue 4: 60-65, DOI: 10.25236/FER.2025.080409

computation, let us assume $i=1$. Through straightforward calculation, we obtain: $A_{11}=-3, A_{12}=-6, A_{13}=-9$. Finally, substituting the independent variables yields the solution of the function, which is the value of the determinant.

$$f(x_{11}, x_{12}, x_{13}, x_{21}, \cdots, x_{33}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

Although these two examples are simple, they demonstrate how to interpret determinants as functions and compute them by assigning specific variable values. This approach remains applicable to more complex determinants, albeit with more intricate forms of functions and variable assignments. Through such examples, we can observe that introducing the concept of functions into the study of determinants helps students better grasp the essence of determinants and their computational methods, while also enhancing their mathematical thinking and problem-solving skills. In teaching and educational research, this method can serve as an effective supplementary tool to assist students in mastering the relevant knowledge of determinants.

3. Conclusions

This study centers on the functional perspective to fundamentally dissect the nature of determinants, systematically unveiling the intrinsic connection between functions and determinants. By employing the method of undetermined coefficients, we rigorously establish a correspondence between the coefficients of the function f and specific algebraic cofactors A_{ij} . Substituting these mathematical relationships back into the original functional expression yields computational results that precisely align with the determinant's value. This innovative methodology not only provides an intuitive analytical framework for interpreting determinant expansion formulas but also deepens students' comprehension of the underlying mathematical mechanisms governing determinant expansion theorems. Furthermore, this research paradigm strengthens the pedagogical foundation of determinant theory while constructing a robust cognitive framework for advanced topics such as Cramer's Rule and Laplace's Theorem[2]. Its demonstrated methodological value significantly enhances both theoretical exploration and practical implementation in mathematical education.

Acknowledgements

This work was supported by the Graduate Education Reform and Quality Enhancement Project of Henan Normal University (XY2023ZX01) and the Graduate Education Reform and Quality Enhancement Project of Henan Province (YJS2024KC32).

References

[1] Department of Mathematics, Tongji University. Linear Algebra [M]. 6th ed. Beijing: Higher Education Press, 2014: 6-7.

[2] Former Algebra Group, Geometry and Algebra Teaching and Research Office, Department of Mathematics, Peking University. Advanced Algebra [M]. 3rd ed. Beijing: Higher Education Press, 2003: 50-102.

[3] Department of Mathematics, Peking University. Advanced Algebra [M]. 5th ed. Beijing: Higher Education Press, 2019.

[4] Liu Guoxin, Xie Chengkang, Liu Hua. Linear Algebra [M]. Beijing: Science Press, 2012: 15-17.

[5] Xu Bin. Re-exploration of the Expansion Mechanism of Determinants [J]. Journal of Science of Teachers College and University, 2020, 40(4): 16-19.

[6] Zhao Ling, Wang Shuquan, Li Fushan. A New Method for Constructing Determinant Theory [J]. Journal of Qufu Normal University (Natural Science Edition), 2024, 50(3): 123-128.

[7] Xia Mengxue. Teaching Design for Linear Algebra Based on Geometric Interpretation: Taking "Properties of Determinants" as an Example [J]. Industrial and Science Tribune, 2025, 24(2): 198-200.

[8] He Chaokui, Zheng Sujuan, Sun Zhongxi. Geometric Interpretation of Determinants and Calculation of Polyhedron Volumes [J]. College Mathematics, 2021, 37(3): 105-109.