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Ground State Solutions for Schrödinger Equations with Periodic Potentials

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Abstract: *In this paper, we study a class of Schrödinger equations including multiple different periodic potentials, this type of equation has a strong physical background and has become a hot topic in current research, especially its widespread application in the theory of Bose-Einstein condensates. Under some appropriate assumptions, we prove the existence of ground state solutions using the variational methods and the concentration compactness principle. Additionally, defining the equation on an unbounded domain and excluding semi-trivial solutions are relatively difficult parts. In the proofs we apply the variant of the Mountain Pass Theorem where it is considered the Gerami condition instead of the Palais-Smale condition.*

Keywords: *Schrödinger equations, Ground state solutions, variational methods*

1. Introduction

This paper deals with the existence of ground state solutions to the following Schrödinger equation

$$\begin{cases} -\Delta u + a_1(x)u = b_1(x)u^3 + \lambda v^2 u, & x \in \mathbf{R}^N, \\ -\Delta v + a_2(x)v = b_2(x)v^3 + \lambda u^2 v, & x \in \mathbf{R}^N, \end{cases} \tag{1}$$

where $N \le 3$, $a_i(x), b_i(x)\,(i = 1,2)$ are positive periodic functions in the variable x , $\lambda > 0$ is a coupling constant. These systems of equations are widely used in various branches of physical problems, such as nonlinear optics, materials science, Bose-Einstein condensates theory and so on. In particular, when $\lambda =$

0, Eq. (1) gives back to two general Schrödinger equations. In the past several years, many authors[1-6] have studied the linear coupled Schrödinger equation

$$\begin{cases} -\Delta u + u = f(u) + \lambda v, & x \in \mathbf{R}^N, \\ -\Delta v + v = g(v) + \lambda u, & x \in \mathbf{R}^N, \end{cases} \tag{2}$$

here $u, v \in H^1(\mathbf{R}^N)$, $f(u), g(v)$ are nonlinear continuous functions. Under certain conditions, they proved the existence of a series of nontrivial solutions for Eq. (2). Furthermore, it is not difficult to see that Eq. (1) is more complex than Eq. (2) due to the different forms of coupling terms. More precisely, Eq. (2) does not have semi-trivial solutions $(u,0)$ and $(0,v)$, while Eq. (1) requires more precise analysis techniques to exclude semi-trivial solutions.

Motivated by the works mentioned above, especially by [3,6], the purpose of this paper is to study the Eq. (1) involving multiple different periodic potentials. More specifically, based on the variational methods, we follow the Mountain Pass Theorem of the Gerami condition instead of the Palais-Smale condition, the Nehari manifold method and the concentration compactness principle to prove the existence of the ground state solutions for Eq. (1). In order to obtain the main results, we assume that $a_i(x), b_i(x)$ and λ satisfying the following conditions:

(V1) $a_i(x), b_i(x) \in C(\mathbf{R}^N, \mathbf{R}) \cap L^\infty(\mathbf{R}^N)$ are 1-periodic in each of $x_1, x_2, \cdots, x_N$;

(V2) there exist constants $\overline{a}_i, \overline{b}_i > 0$ such that $a_i(x) \ge \overline{a}_i, b_i(x) \ge \overline{b}_i$ for all $x \in \mathbf{R}^N$;

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(V3) there exists constant $\hat{b}_i > 0$ such that $\hat{b}_i = \sup_{x\in\mathbf{R}^N} b_i(x)$ and coupling constant $\lambda > 0$.

Now we provide a detailed presentation of our main results.

Theorem 1 Suppose that $(V1)-(V3)$ hold. Then there exists $\hat{\lambda} > 0$ such that $\lambda > \hat{\lambda}$, Eq. (1) has at least one ground state solution $(\bar{u}, \bar{v})$.

2. Preliminaries and Functional Setting

In this section, we introduce some notations and establish the variational setting of Eq. (1) for use in the entire paper.

$|\cdot|_p$ is the usual norm of the space $L^p(\mathbf{R}^N)$ for all $1 \le p \le \infty$, $B_r(x) := \{y \in \mathbf{R}^N : |y-x| < r\}$ for any $r > 0$ and $x \in \mathbf{R}^N$, $o(1)$ denotes any quantity which tends to zero when $n \to \infty$. Moreover, if we take a subsequence of the sequence $\{(u_n, v_n)\}$, we shall denote it again as $\{(u_n, v_n)\}$. Recalling that the definition of the Hilbert space $H^1(\mathbf{R}^N)$ endowed with the standard scalar product and norm

$$\langle u,v\rangle = \int_{\mathbf{R}^N} (\nabla u \nabla v + uv)\mathrm{d}x, \qquad \|u\| = \langle u,u\rangle^{\frac{1}{2}}.$$

Let the working space

$$H_T^1(\mathbf{R}^N) = \{u \in H^1(\mathbf{R}^N) : \|u\|_T^2 < \infty\}$$

be equipped with the norm $\|u\|_T^2 = \int_{\mathbf{R}^N} (|\nabla u|^2 + T|u|^2)\mathrm{d}x$, where $T := a_i(x)$. Moreover, define the norm of the space $E = H^1_{a_1(x)}(\mathbf{R}^N) \times H^1_{a_2(x)}(\mathbf{R}^N)$ as

$$\|(u,v)\| = (\|u\|^2_{a_1(x)} + \|v\|^2_{a_2(x)})^{\frac{1}{2}},$$

and E^* is the dual space of E. We infer from $(V1)-(V3)$ that the norms $\|\cdot\|_{a_i(x)}$ and $\|\cdot\|$ are equivalent. In the meantime $H^1_{a_i(x)}(\mathbf{R}^N)$ is continuously embedded into $L^p(\mathbf{R}^N)$ for all $p \in [2, 2^*]$, where $2^* = \infty$ if $N = 1, 2$, and $2^* = 6$ if $N = 3$.

As a consequence, the functional $J \in C^1(E, \mathbf{R})$ given by

$$J(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{4}\int_{\mathbf{R}^N} (b_1(x)|u|^4 + b_2(x)|v|^4)\mathrm{d}x - \frac{\lambda}{2}\int_{\mathbf{R}^N} |u|^2|v|^2\,\mathrm{d}x$$

is well defined. Thus, we deduce from the critical point theory that every nontrivial critical point of J

is a solution of Eq. (1). Furthermore, the Nehari manifold corresponding to J is defined by

$$\mathcal{N} = \{(u,v) \in E \setminus \{(0,0)\} : \langle J'(u,v), (u,v)\rangle = 0\}$$

and

$$c := \inf_{(u,v)\in\mathcal{N}} J(u,v).$$

3. Proof of Theorem 1

In this section, we give the proof of Theorem 1. Before that, we need to some useful lemmas.

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Lemma 1 Suppose that $(V1)-(V3)$ are satisfied. Then

1) for any $z=(u,v)\in E\setminus\{(0,0)\}$, there is a unique $t_z>0$ such that $t_z z\in\mathcal{N}$;

2) $J(z)\ge J(tz)$ for all $z=(u,v)\in\mathcal{N}$ and $t\ge 0$.

Proof (1) For each $z=(u,v)\in E\setminus\{(0,0)\}$ and $t\ge 0$, we define $\phi(t)=J(tz)$, using $\phi'(t)=0$ leads to $tz\in\mathcal{N}$, which implies that

$$\|(u,v)\|^2=t^2\int_{\mathbf{R}^N}(b_1(x)|u|^4+b_2(x)|v|^4+2\lambda|u|^2|v|^2)\mathrm{d}x. \tag{3}$$

By $(V1)-(V3)$, we have $\phi(0)=0$, $\phi(t)>0$ for $t>0$ small and $\phi(t)<0$ for t large, due to the right end of (3) strictly monotone increasing. Then, $\max_{t\ge0}\phi(t)$ is achieved at a unique $t_z>0$ so that $\phi'(t_z)=0$ and $t_z z\in\mathcal{N}$.

(2) For all $z=(u,v)\in\mathcal{N}$ and $t\ge 0$, one sees that

$$\begin{aligned}J(z)-J(tz)&=\frac{1-t^2}{2}\|z\|^2+\frac{t^4-1}{4}\int_{\mathbf{R}^N}(b_1(x)|u|^4+b_2(x)|v|^4)\mathrm{d}x+\frac{t^4-1}{2}\lambda\int_{\mathbf{R}^N}|u|^2|v|^2\,\mathrm{d}x\\&=\left(\frac{t^2-1}{2}\right)^2\int_{\mathbf{R}^N}(b_1(x)|u|^4+b_2(x)|v|^4+2\lambda|u|^2|v|^2)\mathrm{d}x\\&\ge 0.\end{aligned}$$

Then (2) holds.

Lemma 2 (Mountain Pass Geometry, see[7]) Suppose that $(V1)-(V3)$ are satisfied. It is easy to verify that the functional J satisfies the following conditions:

1) there exist positive constants τ,ρ such that $J(u,v)\ge\tau$ provided with $\|(u,v)\|=\rho$;

2) there exists $(e_1,e_2)\in E$ with $\|(e_1,e_2)\|>\rho$ such that $J(e_1,e_2)<0$.

Define

$$c_1=\inf_{(u,v)\in E\setminus\{(0,0)\}}\max_{t\ge0}J(tu,tv),\quad c_2=\inf_{\gamma\in\Gamma}\sup_{0\le t\le1}J(\gamma(t)),$$

where

$$\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=(0,0),J(\gamma(1))\le 0\}.$$

Consequently, by Lemmas 1 and 2, similar to the proof in [8], one can check that $c=c_1=c_2>0$ and there exists a Gerami sequence $\{(u_n,v_n)\}\subset E$ at the level c such that

$$J(u_n,v_n)\to c\quad\text{and}\quad(1+\|(u_n,v_n)\|_E)\|J'(u_n,v_n)\|_{E^*}\to 0. \tag{4}$$

Lemma 3 Suppose that $(V1)-(V3)$ are satisfied. Then the following results hold:

1) any Gerami sequence $\{(u_n,v_n)\}\subset E$ satisfying (4) is bounded;

2) if (u,v) is a ground state solution of Eq. (1), then there exists a constant $\hat{\lambda}>0$ such that $u,v\ne0$ for any $\lambda>\hat{\lambda}$.

Proof (1) For any Gerami sequence $\{(u_n,v_n)\}$ satisfying (4) in E, we have

$$c+o(1)=J(u_n,v_n)-\frac{1}{4}\langle J'(u_n,v_n),(u_n,v_n)\rangle=\frac{1}{4}\|(u_n,v_n)\|^2,$$

which implies that $\{(u_n,v_n)\}$ is bounded in E;

(2) Following the idea of the proof [9,10], it is easy to see that

$$-\Delta u+a_1(x)u=b_1(x)u^3,\qquad x\in\mathbf{R}^N \tag{5}$$

and

$$-\Delta v+a_2(x)v=b_2(x)v^3,\qquad x\in\mathbf{R}^N \tag{6}$$

have at least ground state solutions u_1 and v_1 respectively in $H^1(\mathbf{R}^N)$. Thus, in order to prove (ii) in Lemma 3, we need to verify the following inequality

$$c<\min\{J(u_1,0),J(0,v_1)\}. \tag{7}$$

According to the discussion method in Lemma 3.3 of [3], one can get that

$$c=\inf_{(u,v)\in E\setminus\{(0,0)\}}\Phi(u,v),$$

where

$$\Phi(u,v)=\frac{\left(\|u\|_{a_1(x)}^2+\|v\|_{a_2(x)}^2\right)^2}{4\int_{\mathbf{R}^N}(b_1(x)|u|^4+b_2(x)|v|^4+2\lambda|u|^2|v|^2)\mathrm{d}x}.$$

Let each $b_i(x)$ be fixed with satisfy (V1) and (V2), define function

$$\begin{aligned}\varphi(s,t)&=\Phi(\sqrt{s}u_1,\sqrt{t}v_1)\\&=\frac{\left(s\|u_1\|_{a_1(x)}^2+t\|v_1\|_{a_2(x)}^2\right)^2}{4\int_{\mathbf{R}^N}(b_1(x)s^2|u_1|^4+b_2(x)t^2|v_1|^4+2\lambda st|u_1|^2|v_1|^2)\mathrm{d}x}\\&:=w(s,t)\end{aligned}$$

on a set $\Lambda:=\{(s,t):s,t\geq 0,(s,t)\neq(0,0)\}$. We noticed that u_1 and v_1 are ground state solutions of (5) and (6) respectively, thus one has

$$\|u_1\|_{a_1(x)}^2=\int_{\mathbf{R}^N}b_1(x)|u_1|^4\,\mathrm{d}x\quad\text{and}\quad\|v_1\|_{a_2(x)}^2=\int_{\mathbf{R}^N}b_2(x)|v_1|^4\,\mathrm{d}x. \tag{8}$$

Furthermore, it is not difficult to obtain

$$w(s,0)=\frac{\|u_1\|_{a_1(x)}^4}{4\int_{\mathbf{R}^N}b_1(x)|u_1|^4\,\mathrm{d}x}=J(u_1,0),$$

$$w(0,t)=\frac{\|v_1\|_{a_2(x)}^4}{4\int_{\mathbf{R}^N}b_2(x)|v_1|^4\,\mathrm{d}x}=J(0,v_1).$$

To prove that (7) holds, it is necessary to prove that $w(s,t)$ does not take the minimum on the lines $s=0$ or $t=0$ of Λ. Thus, for all $P,Q,R,S,T>0$, we can easily verify that the necessary and sufficient condition for binary function

$$g(s,t):=\frac{(Ps+Qt)^2}{Rs^2+2Sst+Tt^2}$$

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not to take a minimum is $QS-PT>0, PS-QR>0$. Then from $QS-PT>0$ that

$$\lambda \| v_1 \|_{a_2(x)}^2 \int_{\mathbf{R}^N} |u_1|^2 |v_1|^2 \, \mathrm{d}x - \| u_1 \|_{a_1(x)}^2 \int_{\mathbf{R}^N} b_2(x) |v_1|^4 \, \mathrm{d}x > 0 . \tag{9}$$

Therefore, by (8) and (9), we have

$$\lambda > \frac{\| u_1 \|_{a_1(x)}^2 \int_{\mathbf{R}^N} b_2(x) |v_1|^4 \, \mathrm{d}x}{\| v_1 \|_{a_2(x)}^2 \int_{\mathbf{R}^N} |u_1|^2 |v_1|^2 \, \mathrm{d}x} = \frac{\int_{\mathbf{R}^N} b_1(x) |u_1|^4 \, \mathrm{d}x}{\int_{\mathbf{R}^N} |u_1|^2 |v_1|^2 \, \mathrm{d}x}$$

$$\geq \frac{\overline{b}_1 \int_{\mathbf{R}^N} |u_1|^4 \, \mathrm{d}x}{\int_{\mathbf{R}^N} |u_1|^2 |v_1|^2 \, \mathrm{d}x} .$$

Similarly, it follows from $PS-QR>0$ and (8) that

$$\lambda > \frac{\overline{b}_2 \int_{\mathbf{R}^N} |v_1|^4 \, \mathrm{d}x}{\int_{\mathbf{R}^N} |u_1|^2 |v_1|^2 \, \mathrm{d}x} .$$

Finally, we define

$$\hat{\lambda} := \max\left\{ \frac{\overline{b}_1 \int_{\mathbf{R}^N} |u_1|^4 \, \mathrm{d}x}{\int_{\mathbf{R}^N} |u_1|^2 |v_1|^2 \, \mathrm{d}x}, \frac{\overline{b}_2 \int_{\mathbf{R}^N} |v_1|^4 \, \mathrm{d}x}{\int_{\mathbf{R}^N} |u_1|^2 |v_1|^2 \, \mathrm{d}x} \right\},$$

then (7) holds when $\lambda > \hat{\lambda}$.

We are Now Ready to Prove Theorem 1 From (i) in Lemma 3, there exists bounded Gerami sequence $\{(u_n, v_n)\} \subset E$ satisfying (4). If

$$\xi = \limsup_{n\to\infty} \sup_{y\in \mathrm{R}^N} \int_{B_1(y)} (|u_n|^2 + |v_n|^2) \, \mathrm{d}x = 0 ,$$

it follows from the Lions' concentration compactness principle [11] that $|u_n|_p^p + |v_n|_p^p \to 0,$ for all $p \in (2, 2^*)$.

Then one has

$$c = J(u_n, v_n) - \frac{1}{2}\langle J'(u_n, v_n), (u_n, v_n) \rangle + o(1)$$

$$= \frac{1}{4}\int_{\mathbf{R}^N} (b_1(x)|u_n|^4 + b_2(x)|v_n|^4 + 2\lambda |u_n|^2 |v_n|^2) \mathrm{d}x + o(1)$$

$$\leq \frac{1}{4}\int_{\mathbf{R}^N} \left[(\hat{b}_1 |u_n|^4 + \hat{b}_2 |v_n|^4 + \lambda(|u_n|^4 + |v_n|^4) \right] \mathrm{d}x + o(1)$$

$$\leq \frac{1}{4}\max\{\hat{b}_1, \hat{b}_2, \lambda\} \int_{\mathbf{R}^N} (|u_n|^4 + |v_n|^4) \mathrm{d}x + o(1)$$

$$= o(1) .$$

This is a contradiction. Thus $\xi > 0$.

Passing to a subsequence if necessary, there exists $k_n \in \mathbf{Z}^N$ such that

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$$\int_{B_{1+\sqrt{N}}(k_n)} (|u_n|^2 + |v_n|^2)\,\mathrm{d}x > \frac{\xi}{2}. \tag{10}$$

Set $(\overline{u}_n(x), \overline{v}_n(x)) = (u_n(x+k_n), v_n(x+k_n))$. We have

$$\int_{B_{1+\sqrt{N}}(0)} (|\overline{u}_n|^2 + |\overline{v}_n|^2)\,\mathrm{d}x > \frac{\xi}{2}.$$

Combining (V1) ,(4) and the translation invariance of J and $\mathcal{N}$, we obtain that

$$J(\overline{u}_n, \overline{v}_n) \to c \quad \text{and} \quad (1 + \|(\overline{u}_n, \overline{v}_n)\|_E) \|J'(\overline{u}_n, \overline{v}_n)\|_{E^*} \to 0. \tag{11}$$

If necessary, take another subsequence, we assume that there exists $(\overline{u}, \overline{v}) \in E$ such that $(\overline{u}_n, \overline{v}_n)$ $\xrightarrow{\text{weakly}} (\overline{u}, \overline{v})$ in E , $(\overline{u}_n, \overline{v}_n) \to (\overline{u}, \overline{v})$ in $L^p_{loc}(\mathbf{R}^N)$ $(\forall\, p \in [2, 2^*))$ and $(\overline{u}_n(x), \overline{v}_n(x)) \to (\overline{u}, \overline{v})$ a.e. on $\mathbf{R}^N$. It follows from (10) that $(\overline{u}, \overline{v}) \neq (0,0)$. Using a standard analysis, one has $J'(\overline{u}, \overline{v}) = 0$, $(\overline{u}, \overline{v}) \in \mathcal{N}$ and $J(\overline{u}, \overline{v}) \geq c$. Furthermore, by (11) and Fatou's lemma, we can see that

$$c = \lim_{n\to\infty}\left[J(\overline{u}_n, \overline{v}_n) - \frac{1}{2}\left\langle J'(\overline{u}_n, \overline{v}_n), (\overline{u}_n, \overline{v}_n)\right\rangle \right]$$

$$= \lim_{n\to\infty} \frac{1}{4}\int_{\mathbf{R}^N} (b_1(x)|\overline{u}_n|^4 + b_2(x)|\overline{v}_n|^4 + 2\lambda|\overline{u}_n|^2|\overline{v}_n|^2)\mathrm{d}x$$

$$\geq \frac{1}{4}\int_{\mathbf{R}^N} (b_1(x)|\overline{u}|^4 + b_2(x)|\overline{v}|^4 + 2\lambda|\overline{u}|^2|\overline{v}|^2)\mathrm{d}x$$

$$= J(\overline{u}, \overline{v}) - \frac{1}{2}\left\langle J'(\overline{u}, \overline{v}), (\overline{u}, \overline{v})\right\rangle$$

$$= J(\overline{u}, \overline{v}).$$

Then we have $J(\overline{u}, \overline{v}) = c$. Recalling that the (ii) of Lemma 3, there exists a constant $\hat{\lambda} > 0$ such that $\overline{u}, \overline{v} \neq 0$ for $\lambda > \hat{\lambda}$, thus we conclude that $(\overline{u}, \overline{v})$ is a ground state solution.

4. Conclusions

Based on the variational methods and the critical point theory, this paper proved the existence of ground state solutions for a class of two-component coupled Schrödinger equations with multiple different periodic potentials. In order to find the ground state solutions, we adopted a series of clever analytical techniques to overcome the difficulties of exclude semi-trivial solutions and lack of compactness.

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