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Chaotic motion of Duffing-Rayleigh oscillator under the Gaussian White Noise and stochastic harmonic excitations

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ABSTRACT. *In this article, we investigate the chaotic behavior of Duffing-Rayleigh oscillator under both the Gaussian White Noise and harmonic excitations. Applying the stochastic Melnikov technique, we obtain the necessary threshold conditions for chaotic motion of this deterministic system theoretically. Simultaneously, by the numerical simulation, the safe basins are introduced to show how the stochastic perturbation affects the safe basin when the Gaussian White Noise amplitude and harmonic excitation increase. The chaotic natures of the sample time series of the system are showed by the Lyapunov exponent and phase portrait maps. The results show that the safe basins appear fractal boundary under both the Gaussian White Noise and harmonic excitations.*

Keywords*: Duffing-Rayleigh oscillator, Gaussian White Noise, harmonic excitation, fractal basin boundary, phase portrait, Lyapunov exponent*

1. Introduction

In 1990s, many people had studied that the stochastic Melnikov method was applied to study the influence of the noise on homoclinic or heteroclinic bifurcation and noise-induced chaos [1-4]. Wiggins [5] found that the system capable of chaotic behavior under harmonic excitation can also behave chaotically if the excitation is quasi-periodic. Effect of bounded noise on the chaotic behavior of the Duffing oscillator under parametric excitation is studied via Melnikov method by W.Y. Liu et al. [6]. Xie [7] used a modified Melnikov function to study the chaotic dynamical behavior for the Duffing oscillator under both harmonic excitation and White Noise

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excitation, and the study found that the weak noise increses the amplitude of the harmonic force. Moon.F. C and Li. G.X [8], by using the Melnikov method and numerical simulation, observed a fractal-looking basin boundary for forced periodic motions of a particle in a two-well potential. J. Peter and H. Peter [9] studied that the effect of external Gaussian White Noise on the invariant measure of a periodically driven damped nonlinear oscillator. F.X. Zhang [10], by using Manasevich–Mawhin continuation theorem and some analysis skill, obtained some sufficient conditions for the existence and uniqueness of periodic solutions for Duffing type *p*-Laplacian differential equation. H.Q. Li et al. [11] gave a mathematically analytical proof on the existence of chaos in a generalized Duffing-type oscillator with fractional-order deflection. So the effect of the Gaussian White Noise or harmonic excitation on the chaos of GoodWin oscillator has been studied recently. Last next we use of Melnikov method analyzing Duffing-Rayleigh osillator.

The equation of the Rayleigh–Duffing oscillator is given by [3]

$$\ddot{x}-\mu(1-\dot{x}^2)\dot{x}-\alpha x+\beta x^3 = f\cos(\omega t), \qquad (1)$$

where, μ is the nonlinear damping coefficient, α and β are respectively the linear and nonlinear parameters, and f and ω are respectively amplitude and frequency of the external force. However, various factors will have some impact on the system in an actual system. M. Siewe Siewe et al. [3, 4] investigated the chaotic behavior of Duffing-Rayleigh oscillator under harmonic external excitation and found that the shape of the basin boundaries of attraction are fractals as the damping increases above the threshold of Melnikov chaos.E. Perkins and B. Balachandran [12] applied this Rayleigh-Duffing oscillator to study noise-enhanced response of nonlinear oscillators, as follows:

$$\ddot{x}-\mu(1-\gamma\dot{x}^2)\dot{x}+x^3 = F\sin(\Omega t)+\sigma\dot{W}(t), \qquad (2)$$

where, $F\sin(\Omega t)$ and $\sigma\dot{W}(t)$ are a superposition of deterministic and stochastic forcing for the system (2), μ γ and Ω are deterministic counterpart of

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the Eq. (2).

However, Xie[13] has focused on the Duffing–Rayleigh oscillator subject to harmonic and stochastic excitations using path integration based on the Gauss–Legendre integration scheme.

In our article, we apply Melnikov method to discuss that the Duffing-Rayleigh oscillator is written as follows when both the harmonic and the stochastic excitations are imposed.

This paper is organized as follows: In Section 2, we use the Melnikov method analyzing and getting the conditions that the chaotic is caused. In section 3, by the numerical simulation, we explain that when we adjust two excitations concurrently that the fractal boundary appears. In Section 4, we use Lyapunov exponents and phase portrait show the chaotic changes when the perturb bation varied little. Finally we summarize our results in Section 5.

2. The Stochastic Melnikov Function

When deterministic and stochastic forcing are stochastic excitation and the Gaussian White Noise separately, letting $\dot{x} = y$, the Duffing-Rayleigh Eq. (2) is rewritten as:

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + \varepsilon(\mu y - \mu y^3 + f_1 \cos(\omega t) + f_2 \xi(t)) \end{cases} , \qquad (3)$$

where, $0 < \varepsilon \ll 1$, $\xi(t)$ is taken as the Gaussian White Noise, μ and f_1 are the damping coefficient, f_2 is the strength of this stochastic excitation. The spectral density for $\xi(t)$ is assumed to be S_0 : $S_0 = \dfrac{1}{2\pi}$.

In order to make use of Melnikov's approach, we need to analyze the equilibrium points and their stability for the unperturbed system. When $\varepsilon = 0$, Eq. (3) is

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considered as unperturbed system, and can be written as follows:

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases}, \tag{4}$$

Eq. (4) is a Hamiltonian system with the Hamiltonian function

$$H(x, y) = \frac{1}{2} y^2 - \frac{1}{2} x^2 + \frac{1}{4} x^4. \tag{5}$$

Since the Hamiltonian is conserved, any level curve (H = constant) is a solutions of Eq. (5). It is easy to show the system possesses a heteroclinic orbit $\Gamma^{\pm}$ to two saddle points $(-1, 0)$ and $(1, 0)$, which is given by

$$\begin{cases} x_0(t) = \pm\sqrt{2} \sec h(t) \\ y_0 = \mp\sqrt{2} \sec h(t) \tanh(t) \end{cases}. \tag{6}$$

The stochastic Melnikov method is a way that is applied to study the nonlinear dynamic system is perturbed. One can easily check that the system (3) satisfies the above assumptions and the hyperbolic axes points of system (3) will not disappear under the harmonic and the random excitations, so the Melnikov approach can also be employed to analyze the stochastic system (3). Melnikov integral is a measure of the random distance between the stable and the unstable manifolds, and the Melnikov function is applied to determine whether the system is a transversal intersects. Finally, we watch whether the systems generate the chaos state of motion under the phase portrait. If the Melnikov function has a simple zero, then the system may be chaotic [14]. So according to [3], we can solve a simple zero to get a threshold. Known from the dynamic system [3], the Melnikov integral for the system (3) can be expressed as:

$$\begin{aligned} M(t_0; \mu, f_1, f_2) &= \int_{-\infty}^{+\infty} y_0(t)[\mu y_0(t) - \mu y_0^3(t) + f_1 \cos(\omega(t + t_0)) + f_2 \xi(t + t_0)]dt \\ &= M_d(t_0; \mu, f_1) + M_r(t_0; f_2) , \end{aligned} \tag{7}$$

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Where, $M_d(t_0;\mu,f_1)=\int_{-\infty}^{+\infty}y_0(t)[\mu y_0(t)-\mu y_0^3(t)+f_1\cos(\omega(t+t_0))]dt$ is the deterministic part of $M(t_0;\mu,f_1,f_2)$, which is the mean value of stochastic Melnikov process, and $M_r(t_0;f_2)=\int_{-\infty}^{+\infty}f_2y_0(t)\xi(t+t_0)dt$ is is the component of the random Melnikov process, due to the Gaussian White Noise excitation.

By the direct integration and residue theory, we get

$$M_d(t_0;\mu,f_1)=-\mu\int_{-\infty}^{+\infty}[\mu y_0^4(t)-y_0^2(t)]dt+f_1\int_{-\infty}^{+\infty}\cos(\omega(t+t_0))dt$$

$$=-\mu(-\int_{-\infty}^{+\infty}2\sec h^2\tanh^2 tdt+\int_{-\infty}^{+\infty}4\sec h^4\tanh^4 tdt)\pm 2\pi e^{-\sqrt{2}\pi\omega/2}\cos\omega t_0/(1+e^{-\sqrt{2}\pi\omega})$$

$$=\mu\left[-\frac{4}{3}\tanh^3 t\Big|_{-\infty}^{+\infty}+(\frac{4}{5}\tanh^5 t\Big|_{-\infty}^{+\infty}-\frac{4}{7}\tanh^7 t\Big|_{-\infty}^{+\infty})\right]\pm\sqrt{2}\pi e^{-\sqrt{2}\pi\omega/2}\cos\omega t_0/(1+e^{-\sqrt{2}\pi\omega})$$

$$=\frac{4}{3}\mu-\frac{16}{35}\mu\pm\sqrt{2}\pi e^{-\sqrt{2}\pi\omega/2}\cos\omega t_0/(1+e^{-\sqrt{2}\pi\omega})$$

$$=\frac{92}{105}\mu\pm\sqrt{2}\pi f_1\omega\sec h\left(\frac{\pi\omega}{2}\right)\sin\omega t_0 \quad . \tag{8}$$

The Gaussian White Noise has Gauss distribution character, so it is decided by the random variable a first moment and second moment. Because the random process $M_r(t_0;f_2)$ is still odd function and is zero mean, $E[\xi(t)]=0$,which implies that

$$E(M_r(t_0;f_2))=E\left[\int_{-\infty}^{+\infty}f_2y_0(t)\xi(t+t_0)dt)\right]=\int_{-\infty}^{+\infty}f_2y_0(t)E(\xi(t+t_0))dt=0. \tag{9}$$

The Gaussian White Noise spectral density is: $S_\xi(\Omega)=S_0$. Therefore, we base on the Wiener-Khintehine theory and Wiener-Khintehine formula, autocorrelation

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function of $M_r(t_0;\, f_2)$ is

$$G_{M_r}(\Omega) = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} M_r(t+t_0)dt_0 = f_2\int_{-\infty}^{+\infty} y_0(t)e^{-i\Omega t}$$

$$= 2\sqrt{2}\pi f_2 \Omega e^{-\pi\Omega/2} i/(1+e^{\pi\Omega}) \quad . \qquad (10)$$

Variance of $M_r(t_0;\, f_2)$ is

$$\sigma_{M_r}^2 = \int_{-\infty}^{+\infty}\left|G_{M_r}(\Omega)\right|^2 S_\xi d\Omega$$

$$\approx \frac{4}{5} f_2^2 \pi^2 S_0 \quad . \qquad (11)$$

Because $E[M_r]=0$, we could take into account simple zero of $M(t_0;\mu,\, f_1,\, f_2)$ form probability or statistics.

$$E[M(t_0;\mu,\, f_1,\, f_2)] = E[M_d(t_0;\mu,\, f_1)] \qquad (12)$$

Then, the Melnikov function (5) can be simplified as

$$M_d(t_0;\mu,\, f_1) = \frac{92}{105}\mu \pm \sqrt{2}\pi f_1 \omega \sec h\left(\frac{\pi\omega}{2}\right)\sin\omega t_0 = 0 \quad , \qquad (13)$$

hence

$$\mu = \pm\frac{105\sqrt{2}\pi f_1 \omega \sec h\left(\frac{\pi\omega}{2}\right)\sin\omega t_0}{92} \quad . \qquad (14)$$

So when Melnikov function appears simple zero under the mean, we get the

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condition

$$\mu \le \left| \frac{105\sqrt{2}\pi f_1 \omega \sec h\left(\frac{\pi\omega}{2}\right)}{92} \right| \tag{15}$$

From an energy point of view, the necessary condition that the stochastic Melnikov function appears chaos under Smale meaning is given by

$$E\left[\mu\int_{-\infty}^{+\infty}(y_0^2 - y_0^4)dt\right] = E^2\left[\mu\int_{-\infty}^{+\infty} f_1\cos(\omega(t+t_0))dt\right] + \sigma_{M_r}^2 . \tag{16}$$

Substituting Eq. (9) and Eq. (11) into Eq. (16), we obtain

$$\left(-\frac{92}{105}\mu\right)^2 \le 2\pi^2 f_1^2 \omega^2 \sec h^2\left(\frac{\pi\omega}{2}\right) + \frac{4}{5}\pi^2 f_2^2 S_0 \quad . \tag{17}$$

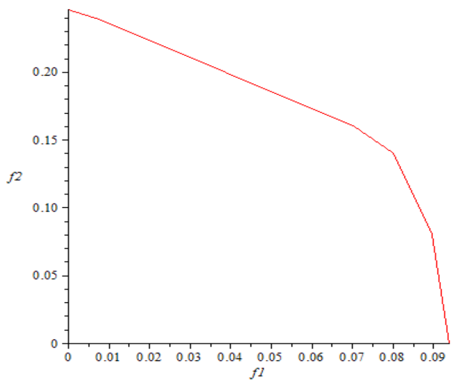

Fig. 1 The relationship of f_1 *and* f_2

Inequality (17) predicts that both Gaussian White Noise excitation and harmonic excitation can induce chaotic response. However, inequality (17) is not sufficient, and other measures are needed to identify the system's chaotic response. In the following two sections, the incursive fractal boundary of safe basin of the system

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and the positive leading Lyapunov exponents are considered. In the present paper, we all set $\varepsilon = 0.1,\quad \omega = 1.0,$ and the step-size $h = 0.001$ unless otherwise indicated.

3. Fractal Boundary of Safe Basin

As is well known, when the system excitations increase the safe basin's area decreases, which the phenomenons are often called basin erosion. The intersecting stable and unstable manifolds have a convoluted structure that extends through a wide region of the phase space. Gan Chunbiao studied only noise-induced or harmonica excitation and the results showed that the boundary of the safe basin can be fractal even if the system is excited only by external Gaussian White Noise, so we study the others situations that the system is excited by both Gaussian White Noise and harmonica excitations .

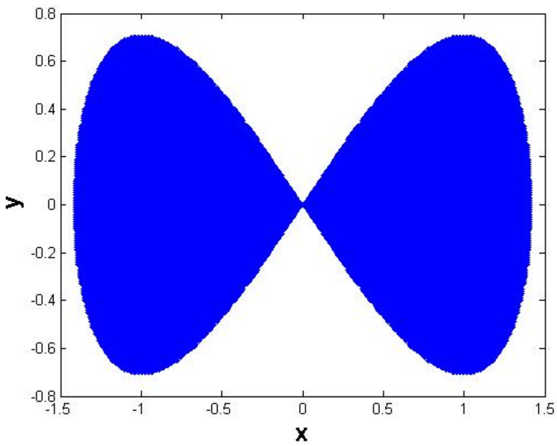

Fig. 2 Safe basin without erosion in the deterministic case of system (2), where $\mu = f_1 = f_2 = 0.0$

□ The safe basin without erosion is drawn within the region G surrounded by the coincident stable and unstable manifolds of the conservative case of system (4) with $\mu = f_1 = f_2 = 0.0$ (see Fig.2). In region G, 300 grid lines in x -direction are

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generated when $-\sqrt{2} \le x \le 0$, while in y-direction, an increasing series of grid lines from 1 to 150 as $-\sqrt{2} \le x \le -\sqrt{2}/2$ and a decreasing series from 149 to 1 as $-\sqrt{2} \le x \le 0$ are sketched. Each grid point is set as an initial condition to perform the simulation of system (4). When the Hamiltonian (6) value of any phase point $(x(t), y(t))$ of the system's trajectory is larger than $H_0 = 0$ within 100000 steps, this motion starting from the corresponding initial point is considered to be unsafe and this initial point is discarded.

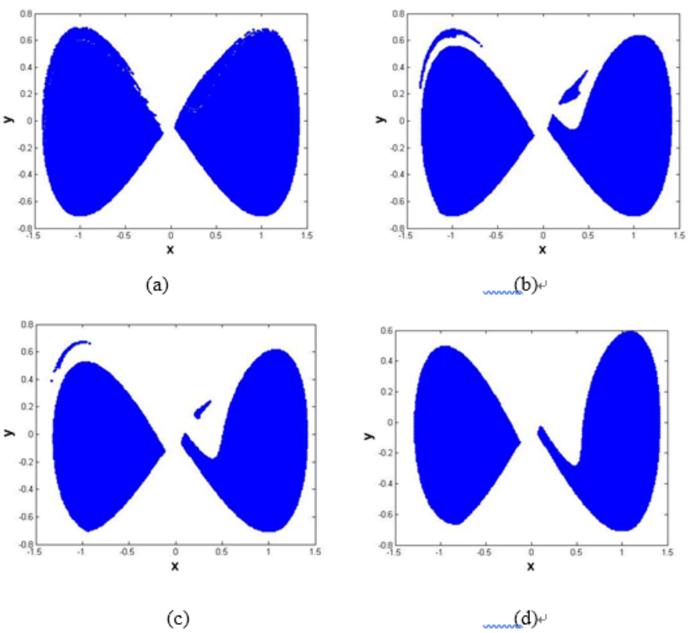

Fig. 3. The effect of the Gaussian White Noise on the boundary of safe basin in system (4) with damping force, where $\mu = -1.0, f_2 = 0.2, G = 0.8. (a) f_1 = 0.55; (b) f_1 = 0.65; (c) f_1 = 0.75 (d) f_1 = 0.85$

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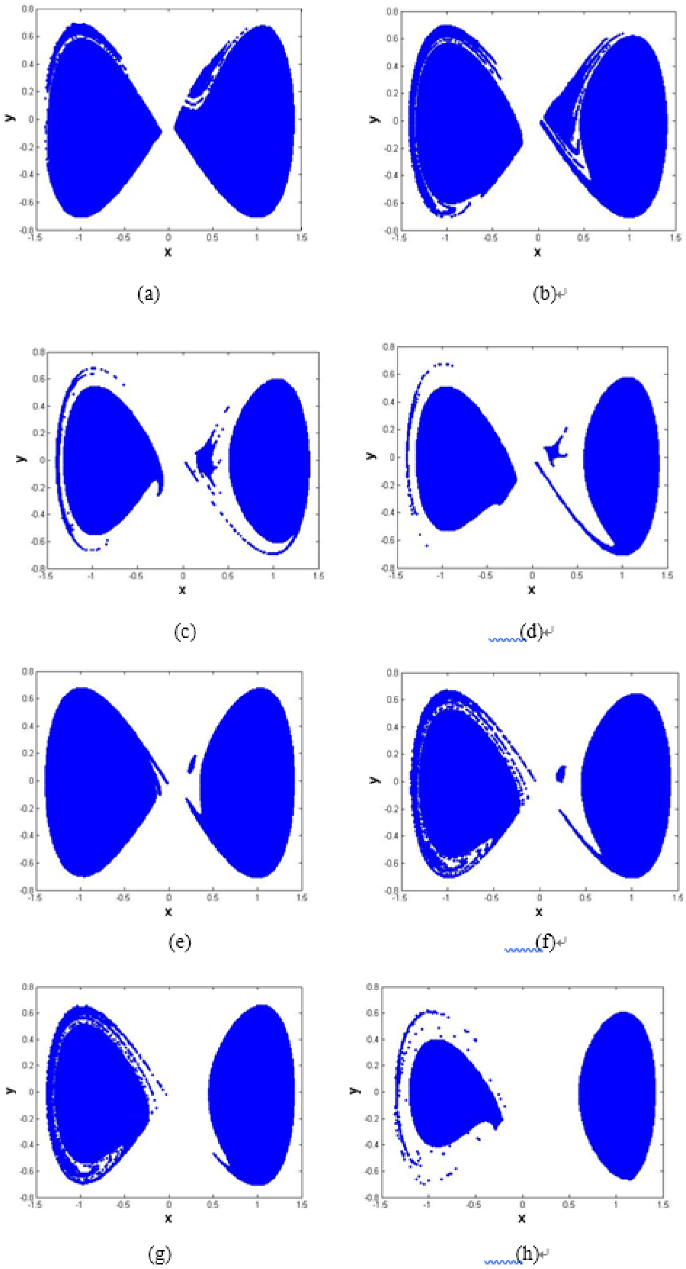

Fig. 4 The boundary of safe basin when the system damping force is excited by

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both the Gaussian White Noise and harmonic excitation, where

$G = 0.8.(a)\mu = -1.0,\ f_1 = 0.55,\ f_2 = 0.25;(b)\mu = -1.0,\ f_1 = 0.65,\ f_2 = 0.35;$

$(c)\mu = -1.0,\ f_1 = 0.75,\ f_2 = 0.45(d)\mu = -1.0,\ f_1 = 0.85,\ f_2 = 0.55;(e)\mu = -0.1,\ f_1 = 0.1,\ f_2 = 0.2;$

$(f)\mu = -0.1,\ f_1 = 0.2,\ f_2 = 0.2;(g)\mu = -0.1,\ f_1 = 0.2,\ f_2 = 0.3;(h)\mu = -0.1,\ f_1 = 0.5,\ f_2 = 0.4;$

In Fig. 2, the safe basin is a closed and apparently smooth curve when $f_1 = 0.0\ \ f_2 = 0.0$. Following the increase of the driving amplitude f_1 and f_2, this orbit will undergo a bifurcation cascade to chaos, and the boundary of the safe basin is fractal (see Fig. 3, in which H is set to be zero). In general, the area of the safe basin decreases following the decrease of the Hamiltonian. To illustrate the effect of the noise and harmonic excitation on the erosion of the safe basin, in Fig. 4(e-h), the pictures present a sequence of safe basins when Gaussian White Noise and harmonic are added to the system. The inclusively fractal images are also observed, which mean that chaos responses exist in the stochastic case of Eq. (4). It is seen that the Gaussian White Noise and harmonic excitation can aggravate the erosion of the system's safe basin. In addition, fractal boundaries can also appear when the Hamiltonian has other positive value.

From Fig. 4, it follows that the response initiating from the potential well does not always escape from the well when the system is excited by both the harmonic excitation and the Gaussian White Noise, and that the boundary of the safe basin can be inclusively fractal. According to inequality (17) (Fig. 3), the Melnikov condition can be satisfied when the strength of the random excitation is large enough, and the chaotic response may arise in the system under the Gaussian White Noise and harmonic excitation. Hence, the boundary of the safe basin may also appear to be inclusively boundary fractal.

5. Analysis of Lyapunov exponents and the phase portrait

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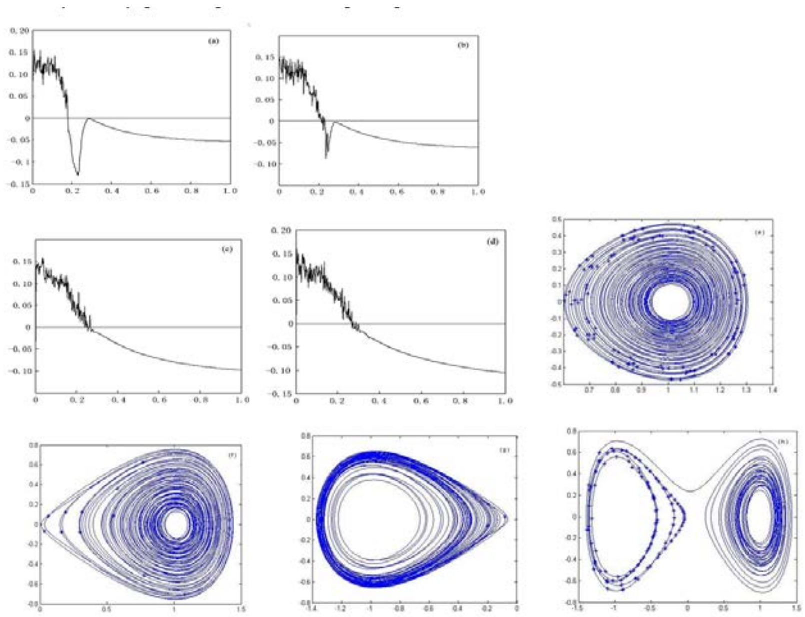

Fig.5.Phase portrait and Lyapunov exponent of system

(4) is excited by both the Gaussian White Noise and harmonic, where: t -direction stands up time, $\mu = -0.1$; $(a). f_1 = 0.1, f_2 = 0.2;$ $(b). f_1 = 0.35, f_2 = 0.4;$ $(c). f_1 = 0.35, f_2 = 0.3;$ $(d). f_1 = 0.5, f_2 = 0.4;$ (e) Phase portrait for the response in (a); (f) phase portrait for the response in (b); (g) phase portrait for the response in (c); (h) phase portrait for the response in (d);

Fig. 5 shows the simulation results for quasi-periodic and chaotic motions in the deterministic case of system (4). In Fig. 5(e-h), the system's responses are both the Gaussian White Noise and harmonic of the phase portrait. These datum are not shown in the figure. This can be learnt from the fact that obvious linear slope appears in the curve of the mean divergence of the response.

We through judge from the simulation results of Fig. 5(a-d), leading Lyapunov exponents' response in the deterministic case may change to be chaotic when both

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the Gaussian White Noise and harmonic excitation are added to the system. With the Wiener process increasing, the chaotic of system (4) will be increasing. After a short transition, there is a long linear region that is used to extract the leading Lyapunov exponent, and the curve will saturate at long times since the system is bounded in phase space. Also, one can easily check that the Melnikov necessary condition (17) is satisfied for the parameter values taken in Fig. 5. Here, the responses in Fig. 5(e) and Fig. 5(g) are called almost-harmonic because, with f_1, f_2 increasing and varying, their leading Lyapunov exponents are approximately zero.

Through leading Lyapunov exponents and phase portrait for the response (4), we know that the chaos is induced by random noise making the system period attractors and chaos attractor of the area diffusions.

6. Summary

By employing the stochastic Melnikov method, the necessary condition for the rising of chaos is obtained, from which one can understand that the almost harmonic, and the chaotic and the thoroughly random responses may exist in the system even though the stochastic Melnikov condition is satisfied. Due to the particularity of the system, the effect of noise and excitation on the boundary of the safe basin is discussed. From the numerical results presented in section 3, the erosion can be aggravated when the driving amplitude of the harmonic excitation and the Gaussian White Noise excitation is increased. The boundary of the safe basin can also become fractal following the increase of the strength of the excitation. When the Gaussian White Noise excitation and harmonic excitation are imposed on the system, the closed loop by the stable and the unstable manifolds in the corresponding conservative system can also be broken down, but will intersect randomly with each other.

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