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Distribution of Last Digits in Integer Sequences

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Abstract: *In this paper, we examine the distribution of the last digits in integer sequences generated by functions of positive integers. Specifically, we investigate the distribution of last digits in squares of integers and the last digits of expressions such as n^2 + n and ax^2 + bx + c, for positive integer values. We derive the theoretical probabilities of these last digits and provide proofs for the distribution properties, including symmetry characteristics.*

Keywords: *last digits; distribution; integer sequences; quadratic forms*

1. Introduction

In number theory, the study of last digits often provides insight into modular behavior and patterns in integer sequences. Our work investigates the statistical distribution of last digits in integer sequences generated by functions of integers, including simple polynomials and quadratic forms. This analysis has applications in cryptography, coding theory, and theoretical computer science, where modular arithmetic and residue patterns are essential.

Notations: We denote Z as the set of integers and $\mathbb{Z}_+$ as the set of positive integers.

2. Main results

Definition 2.1. Let A be a countable set. We define p : A → [0, 1] as the distribution over A, satisfying

$$\sum_{a \in A} p(a) = 1$$

The following theorem characterizes the distribution of the last digit of n^2.

Theorem 2.1 (Distribution of the last digit of n^2). Let n ∈ $\mathbb{Z}_+$. Then {l_1 : last digits of n^2, n ∈ $\mathbb{Z}_+$} = {0, 1, 4, 5, 6, 9} := L_1. The distribution over L_1 is described as p(0) = p(5) = 1/10 and p(1) = p(4) = p(6) = p(9) = 1/20.

Theorem 2.2 (Distribution of the last two digits of n2). Let n ∈ $\mathbb{Z}_+$. Then, {l_2: last two digits of n^2, n ∈$\mathbb{Z}_+$ } = {00, 01, 04, 09, 16, 36, 49, 64, 81, 21, 44, 69, 96, 25, 56, 89, 24, 61, 41, 84, 29, 76} := L_2. The distribution over L_2 as p(00) = p(25) = 1/10 and p(01) = p(04) = p(09) = p(16) = p(36) =p(49) = p(64) = p(81) = p(21) = p(44) = p(69) = p(96) = p(56) = p(89) = p(24) = p(61) =p(41) = p(84) = p(29) = p(76) = 1/25

Corollary 2.3 (the symmetry property about the last two digits of n^2). If n ∈ [0, 49], the last two digits of n^2 exhibit symmetry around the 25^2 = 625.

Theorem 2.4 (Distribution of the last digit of g(t) = t^2 + t. Let t ∈ $\mathbb{Z}_+$.. Then, {l_3 : last digit of [g(x)]^2, n ∈ $\mathbb{Z}_+$= {0, 2, 6} := L_3. The distribution over L_3as p(0) = p(2) = 2/5 and p(6) = 1/5.

Theorem 2.5 (Distribution of the last digit of ax^2 + bx + c). Let n ∈ $\mathbb{Z}_+$Then, {l_4: last digit of [at^2 + bt, n ∈ $\mathbb{Z}_+$} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} := L_4. The distribution over L_4 as p(0) = p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = p(7) = P (8) = p(9) = 1/10

2.1 Proof of Theorem

Proof:Without loss of generality, we can write n = 10m + b, m, b ∈ $\mathbb{Z}_+$.Then we have

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$$n^2 = (10m + b)^2 = 100m^2 + 20mb + b^2.$$

The only thing that affects the last one digit is b^2. In order to find the result of the last digit of b2, we can simply calculate all possible last one digit of b^2 to figure out the last digit of n2. [1]

Before the calculation, let us predict the result first. We can classify b in to two category- b is a even number or b is an odd number. If b is an even number, then we have

$$even * even = even$$

If b is a odd number, then we have

$$odd * odd = odd$$

As a result, 1/2 of b will be even and 1/2 of b will be odd.

$$0^2 = 0$$
$$1^2 = 1$$
$$2^2 = 4$$
$$3^3 = 9$$
$$4^4 = 16$$
$$5^5 = 25$$
$$6^6 = 36$$
$$7^7 = 49$$
$$8^8 = 64$$
$$9^9 = 81$$

We can observe that the frequency of either 0 or 5 occurring as the final digits amounts to 1/10, while the frequency of either 0 or 5 occurring as the final digits amounts to 1/20. Also, 1/2 of the last digit of b2 is even and 1/2 of the last digit of b2 is odd. Therefore, our calculation is correct.

2.2 Proof of Theorem

We cannot let n = 10m + b anymore. This is because when you square the n, as same as above, we have n^2 = (10m + b)^2 = 100m^2 + 20mb + b^2. The 20mb does affect the last two digits of n^2. We will lose the generality if we let n = 10m + b. However, if we consider n = 50m + b, then we have n^2 = (50m + b)^2 = 2500m^2 + 500mb + b^2. As a result, b^2 will represent the last two digits of n^2.

Proof: Without losing any generality, we can let n = 10m + b, m, b $\in \mathbb{Z}_+$. Then, we have

$$n^2 = (50m + b)^2 = 100m^2 + 20mb + b^2$$

The only thing that affects the last two digits is b^2. In order to figure out the last two digits of b^2 without losing generality, rather than let b → [0, 9], there are 49 choices for b to check. Therefore, we need to let b → [0, 49]. we can simply calculate all possible last two digit of b^2 where b → [0, 49] to figure out the last two digits of n2.

Again, before the calculation, let us predict our result. We can classify b in to two category- b is an even number or b is an odd number. If b is a even number, then we have

$$even * even = even$$

If b is a odd number, then we have

$$odd * odd = odd$$

As a result, 1/2 of b will be even and 1/2 of b will be odd.

$$0^2 = 0$$
$$1^2 = 1$$
$$2^2 = 4$$
$$3^3 = 9$$
$$4^4 = 16$$
$$5^5 = 25$$
$$6^6 = 36$$
$$7^7 = 49$$
$$8^8 = 64$$
$$9^9 = 81$$
$$10^2 = 100$$

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$11^2 = 121$
$12^2 = 144$
$13^2 = 169$
$14^2 = 196$
$15^2 = 225$
$16^2 = 256$
$17^2 = 289$
$18^2 = 324$
$19^2 = 361$
$20^2 = 400$
$21^2 = 441$
$22^2 = 484$
$23^2 = 529$
$24^2 = 576$
$25^2 = 625$
$26^2 = 676$
$27^2 = 729$
$28^2 = 784$
$29^2 = 841$
$30^2 = 900$
$31^2 = 961$
$32^2 = 1024$
$33^2 = 1089$
$34^2 = 1156$
$35^2 = 1225$
$36^2 = 1296$
$37^2 = 1369$
$38^2 = 1444$
$39^2 = 1521$
$40^2 = 1600$
$41^2 = 1681$
$42^2 = 1764$
$43^2 = 1849$
$44^2 = 1936$
$45^2 = 2025$
$46^2 = 2116$
$47^2 = 2209$
$48^2 = 2304$
$49^2 = 2401$

Therefore, we proof that the last two digits will only be 00,01,04,09,16,36,49,64,81, 21,44,69,96,25, 56,89,24,61,41,84,29,76, where the frequency of either 00 and 25 occurring as last two digits amounts to 1/10, while the frequency of other numbers occurring as last two digits amounts to 1/20. In addition, 1/2 of the last digit of b2 is even and 1/2 of the last digit of b2 is odd. Therefore, our calculation is correct.

2.3 Proof of Corollary

Proof: Without losing generality, for b in range [0, 50], it is sufficient to consider the numbers in form 50n + (25 + b) for b ∈ [26, 50] and in form 50n + (25 − b) for b ∈ [0, 24]. Let square it first in order to find why that pattern happen. Therefore, we have

$$[50n + (25 + b)]^2 = 5000n + 100nb + 625 - 50b + b^2$$

and

$$[50n + (25 - b)]^2 = 5000n - 100nb + 625 - 50b + b^2$$

We can find that those two numbers have some last two digits—not three because of the +100nb for the square of 50n + (25 + b) and −100nb for the square of 50n + (25 − b). If the last two digits of numbers b ∈ [26, 50] are the same as the last two digits of numbers b ∈ [0, 24], it is obvious that the last two digits of a number's square will symmetry along the 252 = 625.

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2.4 Proof of Theorem

Above all, we try to figure out the last one digit and last two digits of n^2. Let us Change n^2 into difference form, let say, f (x) = x^2 for example. However, what about the last digit of g(t) = t^2 + t?

Proof: Without losing generality, it is sufficient to consider t = 10w + d, where t, w, d $\in \mathbb{Z}_{+}$.

Therefore, we have

$$g(t) = t^2 + t = (10w + d)^2 + (10w + d) = d^2 + 20dw + 100w^2 + d + 10w$$

The only thing that affects the last digit of g(t) is d^2 + d. Without losing generality, there are 10 choice for d to check. We can simply calculate all possible last digit of number d in order to figure out the last digit of g(t).

Before the calculation, let us predict the result first. We can classify d in to two category-d is a even number or b is an odd number If d is an even number, then we have

$$even * even + even = even + even = even$$

If d is an odd number, then we have

$$odd * odd + odd = odd + odd = even$$

Therefore, the last digit of d^2 + d will only be an even number.

$$0^2 + 0 = 0$$
$$1^2 + 1 = 2$$
$$2^2 + 2 = 6$$
$$3^2 + 3 = 12$$
$$4^2 + 4 = 20$$
$$5^2 + 5 = 30$$
$$6^2 + 6 = 42$$
$$7^2 + 7 = 56$$
$$8^2 + 8 = 72$$
$$9^2 + 9 = 90$$

We can observe that the last digit of g(t) will be and only be 0,2,6, where the frequency of 0 and 2 occurring as the final digits amounts to 2/5, while 6 occurring as the final digits amounts to 1/5. Moreover, 0, 2, 6 are even number, which are correspondent with our prediction. Theretofore, our calculation is correct.

2.5 Proof of Theorem

Above all, we discuss the last digit of the t^2 + t. Then, what about the last digit of at^2 + bt + c? At this point, we do not need to consider the +c because we just need to +c in order to get the last one digit of at^2 + bt. Therefore, we now only need to consider the at^2 + bt.

Proof: Without losing generality, it is sufficient to consider t = 10w + d, where t, w, d $\in \mathbb{Z}_{+}$, and a, b→ [0, 49]. Therefore, we have

$$a(10w + d)^2 + b(10w + d) = 100aw^2 + 20adw + ad^2 + 10bw + bd$$

The only thing that will affect the last digit is ad2 + bd. Let calculate all possible result of ad^2 + bd. However, there are a lot of data for the last digit of ad^2+bd. Therefore we will only find the number of distinct final digits for each (a,b) pair.

6	3	6	3	6	3	6	3	6	3
3	6	3	6	3	6	3	6	3	6
6	3	6	3	6	3	6	3	6	3
3	6	3	6	3	6	3	6	3	6
2	5	10	5	10	1	10	5	10	5
3	6	3	6	3	6	3	6	3	6
6	3	6	3	6	3	6	3	6	3
3	6	3	6	3	6	3	6	3	6
6	3	6	3	6	3	6	3	6	3

*first row is a=1, second a=2 ... first column is b=0, second is b=1...

On average, 3.5 distinct final digits occur for a random (a,b) pair. The variance in the number of distinct digits is approximately 4.31, and the distribution does not resemble a Gaussian distribution. Each

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final digit (0 to 9) appears exactly 90 times across all (a,b,d) combinations. The mean is 90, and the variance is 0 because each digit occurs with equal frequency.

3. Conclusion

In this work, we explored the statistical distribution of the last digits in various integer se- quences, particularly focusing on quadratic forms such as n^2, t^2 + t, and ax^2 + bx + c. These results not only provide theoretical value in understanding modular behaviors but also have practical implications in fields like cryptography and coding theory, where residue patterns and modular distributions are essential. Future work can extend these investigations to higher- degree polynomials, explore multidimensional residue distributions, or analyze the implications for random sequence generation and pseudo-random number theory.[2]

By systematically characterizing these distributions, it contribute to a deeper understanding of number theory's role in modern mathematical and computational applications.[3]

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