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# Existence of positive solutions for boundary value problems of nonlinear fractional functional integro-differential equations

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***Abstract:*** *In this paper, we study the existence of positive solutions for a class of fractional functional integro-differential equations with two fractional derivative terms. First, we transform the boundary value problem into an equivalent integral equation, establish the operator T and prove its full continuity, then the existence theorems of positive solutions of boundary value problems is established by using the fixed point theorems of cone extension and cone compression.*

***Keywords:*** *Caputo fractional derivatives, functional integro-differential equations, fixed point theorem, positive solutions*

## 1. Introduction

In recent years, with the further development of the theory of fractional calculus and the application of fractional differential equations in the fields of automatic control theory, biology and viscoelasticity[1-2], which has received extensive attention. Most of the mathematical models established to solve practical problems are completed in an ideal state. It is assumed that the changing laws of things are only related to the current state. However, in actual problems, the future behavior of the system depends not only on the current state. At the same time, it may also be affected by the past state or the rate of state change. Therefore, it is considered that the functional differential equation can describe the objective world more accurately, and it has a wide range of applications in signal recognition, economics, physics and other fields[3].

In this paper, we use the fixed point theorem of cone extension and cone compression to study the following nonlinear fractional functional integro-differential equation boundary value problems with two fractional derivative terms

$$\begin{cases} {}^cD_{0^+}^{\alpha}[{}^cD_{0^+}^{\beta}u(t)+h(t,u_t)]+f(t,u_t,Qu(t))=0, t\in(0,1), \\ u(t)=\varphi(t), \qquad t\in[-\tau,0], \\ {}^cD_{0^+}^{\beta}u(0)=-a, \quad u''(0)=-g(u(\xi)), \\ u(1)=\int_0^1 k(s,u(s))\mathrm{d}s, \end{cases} \tag{1}$$

where ${}^cD_{0^+}^{\alpha}$ is the Caputo fractional derivative operator, $0<\alpha\le 1$, $2<\beta\le 3$, $0<\xi<1$, $\tau>0$, $u_t=u_t(\theta)=u(t+\theta)$, $\theta\in[-\tau,0]$, $g\in C(\mathbb{R}^+,\mathbb{R}^+)$, $C_1=C([-\tau,0],\mathbb{R})$, $\varphi\in C([-\tau,0],\mathbb{R}^+)$, $\varphi(0)=0$, $q_0=\sup_{0\le t\le 1}\int_0^t q(t,s)\mathrm{d}s$,

$q\in C([0,1]\times[0,1],\mathbb{R}^+)$, $C_1^+=\{\psi\in C_1:\psi(\theta)\ge 0,\theta\in[-\tau,0]\}$, $a\in\mathbb{R}^+$,

$a\ge h(0,\varphi)$, $f\in C([0,1]\times C_1^+\times\mathbb{R}^+,\mathbb{R}^+)$, $Qu(t)=\int_0^t q(t,s)u(s)\mathrm{d}s$,

$h\in C([0,1]\times C_1^+,\mathbb{R}^+)$, $k\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$, $\mathbb{R}^+=[0,+\infty)$.

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For any $\psi \in C_1$, we define norm $\|\psi\|_{C_1} = \sup_{\theta\in[-\tau,0]} |\psi(\theta)|$, then $C_1$ is a Banach space. We denote that $E = \{u \in C[-\tau,1] : u(0) = 0\}$ and endowed with the norm $\|u\|_E = \sup_{t\in[-\tau,1]} |u(t)|$, then $(E, \|\cdot\|)$ is a Banach space. And we also denote that $E_0 = \{z \in C[-\tau,1] : z(t) = 0, t \in [-\tau,0]\}$ and endowed with the norm $\|z\| =$

$\sup_{t\in[-\tau,1]} |z(t)| = \sup_{t\in[0,1]} |z(t)|$, then $E_0$ is a Banach space, $E_0$ is a subset of $E$.

## 2. Preliminaries

**Lemma 2.1** If the function $u$ is the solution of the boundary value problem (1.1), then the function $u$ satisfies the following integral equation

$$u(t) = \begin{cases} \int_0^1 G(t,s) f(s,u_s,Qu(s))\mathrm{d}s + \int_0^1 H(t,s)h(s,u_s)\mathrm{d}s + (a - h(0,\varphi)) \times \\ \dfrac{t - t^\beta}{\Gamma(\beta+1)} + t\int_0^1 k(s,u(s))\mathrm{d}s + (t - t^2)\dfrac{1}{2} g(u(\xi)), & t \in (0,1), \\ \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha+\beta)} \begin{cases} t(1-s)^{\alpha+\beta-1} - (t-s)^{\alpha+\beta-1}, & 0 \le s \le t \le 1, \\ t(1-s)^{\alpha+\beta-1}, & 0 \le t \le s \le 1, \end{cases} \tag{3}$$

$$H(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ t(1-s)^{\beta-1}, & 0 \le t \le s \le 1. \end{cases} \tag{4}$$

**Proof:** Let $u(t)$ be the solution to the boundary value problem (1.1), then for any $t \in [0,1]$, the general solution of fractional differential equation

$${}^cD_{0^+}^\alpha [{}^cD_{0^+}^\beta u(t) + h(t,u_t)] + f(t,u_t,Qu(t)) = 0$$

is given by ${}^cD_{0^+}^\beta u(t) = -\dfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u_s,Qu(s))\mathrm{d}s - h(t,u_t) + c_0$. The boundary condition ${}^cD_{0^+}^\beta u(0) = -a$, we can obtain that $c_0 = h(0,\varphi) - a$, so

$$u(t) = -\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,u_s,Qu(s))\mathrm{d}s - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} h(s,u_s)\mathrm{d}s$$
$$+ (h(0,\varphi) - a)\frac{t^\beta}{\Gamma(\beta+1)} + c_1 + c_2 t + c_3 t^2.$$

From the boundary condition $u(0) = \varphi(0) = 0$ implies that $c_1 = 0$. Thus,

$$u(t) = -\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,u_s,Qu(s))\mathrm{d}s - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} h(s,u_s)\mathrm{d}s$$
$$+ (h(0,\varphi) - a)\frac{t^\beta}{\Gamma(\beta+1)} + c_2 t + c_3 t^2.$$

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$$u''(t) = -\frac{1}{\Gamma(\alpha+\beta-2)}\int_0^t (t-s)^{\alpha+\beta-3} f(s,u_s,Qu(s))\mathrm{d}s - \frac{1}{\Gamma(\beta-2)}\times$$
$$\int_0^t (t-s)^{\beta-3} h(s,u_s)\mathrm{d}s + (h(0,\varphi)-a)\frac{t^{\beta-2}}{\Gamma(\beta-1)} + 2c_3.$$

The boundary condition $u''(0) = -g(u(\xi))$, we can obtain that $c_3 = -\frac{g(u(\xi))}{2}$. Thus,

$$u(t) = \frac{-1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,u_s,Qu(s))\mathrm{d}s - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\times$$
$$h(s,u_s)\mathrm{d}s + (h(0,\varphi)-a)\frac{t^{\beta}}{\Gamma(\beta+1)} + c_2 t - \frac{1}{2}t^2 g(u(\xi)).$$

From the boundary condition $u(1) = \int_0^1 k(s,u(s))\mathrm{d}s$, thus,

$$c_2 = \frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} f(s,u_s,Qu(s))\mathrm{d}s + \frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} h(s,u_s)\mathrm{d}s$$
$$-(h(0,\varphi)-a)\frac{1}{\Gamma(\beta+1)} + \frac{1}{2}g(u(\xi)) + \int_0^1 k(s,u(s))\mathrm{d}s.$$

Thus we can get that

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,u_s,Qu(s))\mathrm{d}s - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} h(s,u_s)\mathrm{d}s \\ &\quad + \frac{t}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} f(s,u_s,Qu(s))\mathrm{d}s + \frac{t}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} h(s,u_s)\mathrm{d}s \\ &\quad + \frac{t-t^{\beta}}{\Gamma(\beta+1)}(a-h(0,\varphi)) + (t-t^2)\frac{1}{2}g(u(\xi)) + t\int_0^1 k(s,u(s))\mathrm{d}s \\ &= \int_0^1 G(t,s) f(s,u_s,Qu(s))\mathrm{d}s + \int_0^1 H(t,s)h(s,u_s)\mathrm{d}s + t\int_0^1 k(s,u(s))\mathrm{d}s \\ &\quad + \frac{t-t^{\beta}}{\Gamma(\beta+1)}(a-h(0,\varphi)) + (t-t^2)\frac{1}{2}g(u(\xi)). \end{aligned}$$

When $t\in[-\tau,0]$, $u(t)=\varphi(t)$. So

$$u(t) = \begin{cases} \int_0^1 G(t,s) f(s,u_s,Qu(s))\mathrm{d}s + \int_0^1 H(t,s)h(s,u_s)\mathrm{d}s + t\int_0^1 k(s,u(s))\mathrm{d}s \\ \quad + \frac{t-t^{\beta}}{\Gamma(\beta+1)}(a-h(0,\varphi)) + (t-t^2)\frac{1}{2}g(u(\xi)), & t\in(0,1), \\ \varphi(t), & t\in[-\tau,0]. \end{cases}$$

On the contrary, if (2.1) holds, it is easy to prove that $u(t)$ is the solution of boundary value problem (1.1).

**Lemma2.2** For all $\eta,\omega\in(0,1)$ and $\omega^{\beta-1} < \eta < \omega$, then

(1) $G(t,s)$ is continuous and $0 \le G(t,s) \le p_1(s)$ for $t,s\in[0,1]$; $G(t,s) \ge (\eta-\omega^{\beta-1})p_1(s)$ for $t\in[\eta,\omega]$, $s\in[0,1]$;

(2) $H(t,s)$ is continuous and $0 \le H(t,s) \le p_2(s)$ for $t,s\in[0,1]$; $H(t,s)$

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$\geq (\eta - \omega^{\beta-1}) p_2(s)$ for $t \in [\eta, \omega], s \in [0,1]$;

Where $p_1(s) = \dfrac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}$, $p_2(s) = \dfrac{(1-s)^{\beta-1}}{\Gamma(\beta)}$.

**3. Main results**

Supplement the definition of function $\varphi(t)$, let $t \in [0,1]$, $\varphi(t) = 0$, then $\varphi \in E$. Making a transformation $u(t) = \varphi(t) + z(t)$, then for any $t \in [0,1]$, it is easy to show that $u_t = \varphi_t + z_t$ and the integral equation (2) equivalent to the integral equation

$$z(t) = \begin{cases} \int_0^1 G(t,s) f(s, \varphi_s + z_s, Qz(s)) \mathrm{d}s + \int_0^1 H(t,s) h(s, \varphi_s + z_s) \mathrm{d}s \\ +t\int_0^1 k(s, z(s)) \mathrm{d}s + \dfrac{(t-t^\beta)(a-h(0,\varphi))}{\Gamma(\beta+1)} + \dfrac{(t-t^2)}{2} g(z(\xi)), t \in (0,1), \\ 0, \qquad t \in [-\tau, 0]. \end{cases} \quad (5)$$

Let $P_\omega = \left\{ z \in E_0 : z(t) \geq 0, t \in [-\tau, 1], \min\limits_{t\in[\eta,\omega]} z(t) \geq (\eta - \omega^{\beta-1}) \| z \| \right\}$,

where $\omega \in (0,1), \omega^{\beta-1} < \eta < \omega$. Obviously, $P_\omega \subseteq E_0$ is a cone, which is for any $x, y \in E_0, x \preceq y$ if and only $y - x \in P_\omega$. Then $(E_0, \preceq)$ is a semi-ordered Banach space. We define operator $T : P_\omega \to E_0$ as

$$Tz(t) = \begin{cases} \int_0^1 G(t,s) f(s, \varphi_s + z_s, Qz(s)) \mathrm{d}s + \int_0^1 H(t,s) h(s, \varphi_s + z_s) \mathrm{d}s \\ +t\int_0^1 k(s, z(s)) \mathrm{d}s + \dfrac{(t-t^\beta)(a-h(0,\varphi))}{\Gamma(\beta+1)} + \dfrac{(t-t^2)}{2} g(z(\xi)), t \in (0,1), \\ 0, \qquad t \in [-\tau, 0]. \end{cases} \quad (6)$$

**Lemma 3.1** Assume that $f, h$ satisfies the $f \in C([0,1] \times C_1^+ \times \mathbb{R}^+, \mathbb{R}^+)$,

$h \in C([0,1] \times C_1^+, \mathbb{R}^+)$ conditions, then the operator $T : P_\omega \to P_\omega$ is completely continuous.

**Lemma 3.2** (See[1]) Let $E$ be a Banach space, and let $P \subset E$ be a cone in $E$. Assume $\Omega_1, \Omega_2 \subset P$ are two bounded open subsets of $E$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either

(1) $\| Tz \| \leq \| z \|, \forall z \in P \cap \partial\Omega_1$; $| Tz \| \geq \| z \|, \forall z \in P \cap \partial\Omega_2$;

(2) $\| Tz \| \geq \| z \|, \forall z \in P \cap \partial\Omega_1$; $| Tz \| \leq \| z \|, \forall z \in P \cap \partial\Omega_2$.

Then $T$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

For convenience, we denote

$$f^0 = \limsup_{\|\psi\|_{C_1} + v \to 0^+} \sup_{t\in[0,1]} \frac{f(t,\psi,v)}{\|\psi\|_{C_1} + v}, \ h^0 = \limsup_{\|\psi\|_{C_1} \to 0^+} \sup_{t\in[0,1]} \frac{h(t,\psi)}{\|\psi\|_{C_1}},$$

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$$k^0=\limsup_{v\to 0^+}\sup_{t\in[0,1]}\frac{k(t,v)}{v},\ g^0=\limsup_{v\to 0^+}\frac{g(v)}{v},\ k_\infty=\liminf_{v\to+\infty}\inf_{t\in[0,1]}\frac{k(t,v)}{v}.$$

**Theorem 3.1** Suppose there are constants $N_1,N_2,N_3,N_4,N_5>0$, so that $f^0<N_1, h^0<N_2, k^0<N_3, g^0<N_4,\ k_\infty>N_5$ are established. If $\varphi(\theta)\equiv 0$,

$$\frac{N_1(1+q_0)}{\Gamma(\alpha+\beta+1)}+\frac{N_2}{\Gamma(\beta+1)}+N_3+N_4<1,\quad N_5\ge 1,$$

then there is a constant $b>0$, when $\frac{a-h(0,\varphi)}{\Gamma(\beta+1)}<b$, the boundary value problem (1.1) has at least one positive solution.

**Proof:** Selection cone $P_\omega,\ 0<\omega<1,\ \omega^{\beta-1}<\eta<\omega$, then by Lemma3.1, $T:P_\omega\to P_\omega$ is completely continuous.

We can obtain $\varphi_t=0,\ t\in[0,1]$ from $\varphi(\theta)\equiv 0,\theta\in[-\tau,0]$. Due to $f^0<\ N_1$, then there exists a constant $r_1>0$ such that $f(t,\psi,v)\le N_1(\|\psi\|_{C_1}+v)$ for $\psi\in C_1^+, v\in\mathbb{R}^+$ and $\|\psi\|_{C_1}+v\in[0,r_1]$. In the same way, because of $h^0<N_2$, then there exists a constant $r_2>0$ such that $h(t,\psi)\le N_2\|\psi\|_{C_1}$ for $t\in[0,1]$, $\psi\in C_1^+$ and $\|\psi\|_{C_1}\in[0,r_2]$. Because of $k^0<N_3$, then there exists a constant $r_3>0$ such that $k(t,v)\le N_3 v$ for $t\in[0,1]$ and $v\in[0,r_3]$. Due to $g^0<N_4$, then there exists a constant $r_4>0$ such that $g(v)\le N_4 v$ for $v\in[0,r_4]$.

For convenience, we denote

$$\gamma_1=\frac{N_1(1+q_0)}{\Gamma(\alpha+\beta+1)}+\frac{N_2}{\Gamma(\beta+1)}+N_3+N_4.$$

Let

$$r_5=\min\left\{\frac{r_1}{1+q_0},r_2,r_3,r_4\right\},\quad b=(1-\gamma_1)r_5.$$

According to Lemma2.2, when $0\le\frac{a-h(0,\varphi)}{\Gamma(\beta+1)}\le b$, for any

$$\Omega_{r_5}=\{z\in P_\omega:\|z\|\le r_5\},\ z\in\partial\Omega_{r_5},$$

then $\|z\|=r_5$, we can get that

$$\|\varphi_s+z_s\|_{C_1}=\|z_s\|_{C_1}\le\|z\|=r_5,$$

$$Qz(s)=\int_0^s q(s,\kappa)z(\kappa)\mathrm{d}\kappa\le r_5\int_0^s q(s,\kappa)\mathrm{d}\kappa\le q_0 r_5$$

for $s\in[0,1]$, so $\varphi_s+z_s+Qz(s)\le r_5+q_0r_5\le r_1$. Then, we have

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$$\begin{aligned}
\|Tz\| &= \sup_{t\in[0,1]}\Big(\int_0^1 G(t,s)f(s,\varphi_s+z_s,Qz(s))\mathrm{d}s+\int_0^1 H(t,s)h(s,\varphi_s+z_s)\mathrm{d}s \\
&\quad +t\int_0^1 k(s,z(s))\mathrm{d}s+\frac{t-t^\beta}{\Gamma(\beta+1)}(a-h(0,\varphi))+(t-t^2)\frac{1}{2}g(z(\xi))\Big) \\
&\le \sup_{t\in[0,1]}\Big(\int_0^1 G(t,s)N_1(\|\varphi_s+z_s\|_{C_1}+q_0\|z\|)\mathrm{d}s+N_3\|z\|+N_4\|z\| \\
&\quad +\int_0^1 H(t,s)N_2\|\varphi_s+z_s\|_{C_1}\mathrm{d}s \\
&\le N_1(1+q_0)\|z\|\int_0^1 p_1(s)\mathrm{d}s+N_2\|z\|\int_0^1 p_2(s)\mathrm{d}s+N_3\|z\| \\
&\quad +N_4\|z\|+\frac{a-h(0,\varphi)}{\Gamma(\beta+1)} \\
&= \frac{N_1(1+q_0)}{\Gamma(\alpha+\beta+1)}\|z\|+\frac{N_2}{\Gamma(\beta+1)}\|z\|+N_3\|z\|+N_4\|z\|+\frac{a-h(0,\varphi)}{\Gamma(\beta+1)} \\
&= \Big(\frac{N_1(1+q_0)}{\Gamma(\alpha+\beta+1)}+\frac{N_2}{\Gamma(\beta+1)}+N_3+N_4\Big)\|z\|+\frac{a-h(0,\varphi)}{\Gamma(\beta+1)} \\
&\le \gamma_1 r_5+b=r_5,
\end{aligned}$$

So, for any $z\in P_\omega\cap\partial\Omega_{r_5}$, we get $\|Tz\|\le\|z\|$.

Because of $k_\infty>N_5$, then there exists a constant $r_6>r_5$ such that $k(t,v)\ge N_5 v$ for $t\in[0,1]$ and $v\in[0,(\eta-\omega^{\beta-1})r_6]$. For any $\Omega_{r_6}=\{z\in P_\omega:\|z\|\le r_6\}$, $z\in\partial\Omega_{r_6}$, we have $\|z\|=r_6$. Then

$$\begin{aligned}
\|Tz\| &= \sup_{t\in[0,1]}\Big(\int_0^1 G(t,s)f(s,\varphi_s+z_s,Qz(s))\mathrm{d}s+\int_0^1 H(t,s)h(s,\varphi_s+z_s)\mathrm{d}s \\
&\quad +t\int_0^1 k(s,z(s))\mathrm{d}s+\frac{t-t^\beta}{\Gamma(\beta+1)}(a-h(0,\varphi))+(t-t^2)\frac{1}{2}g(z(\xi))\Big) \\
&\ge \int_0^1 k(s,z(s))\mathrm{d}s\ge N_5\|z\|\ge r_6,
\end{aligned}$$

hence, for any $z\in P_\omega\cap\partial\Omega_{r_6}$, we get $\|Tz\|\ge\|z\|$.

In summary, from Lemma3.2, $T$ has at least one fixed point $z$ in $P_\omega\cap(\overline{\Omega}_{r_6}\setminus\Omega_{r_5})$, and $0<r_5\le\|z\|\le r_6$, so the boundary value problem (1.1) has at least one positive solution.

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