ISSN 2616-5775 Vol. 2, Issue 1: 92-101, DOI: 10.25236/AJCIS.010022

# A Half-space Projection Method for Solving Sum of Two Maximal Monotone Operators

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***ABSTRACT.*** *In this paper, we propose a modified Forward-Backward splitting method for finding a zero of the sum of two operators. A classical modification of Forward-Backward method was proposed by Tseng, which is known to converge when the forward and the backward operators are monotone and with Lipschitz continuity of the backward operator. The algorithm proposed here improves Tseng's method in some instances. The first and main part of our approach, contains an explicit Armijo-type search in the spirit of the extragradient-like methods for variational inequalities. During the iteration process the search performs only one calculation of the forward-backward operator, in each tentative of the step. This achieves a considerable computational saving when the forward-backward operator is computationally expensive. The second part of the scheme consists in special projection steps. The convergence analysis of the proposed scheme is given assuming monotonicity on both operators, without Lipschitz continuity assumption on the backward operator.*

***KEYWORDS:*** *Armijo-type search, Projection method, Half-space, Maximal monotone operators*

## 1. Introduction

In this paper, we present a modified method for solving monotone inclusion problems for the sum of two operators. Given the monotone operators A : domA ⊆ Rn → Rn point-to-point and B : domB ⊆ Rn ⇒ Rn point-to-set, the inclusion problem consists in:

$$\text{Find } x \in R^n \text{ such that } 0 \in (A+B)(x). \tag{1}$$

This problem has recently received a lot attention due to the fact that many nonlinear problems, arising within applied areas, are mathematically modeled as nonlinear operator equations and/or inclusions, which are decomposed as the sum of two operators. We focus our attention in the called splitting method, which is an

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iterative method, for which each iteration involves only the individual operators, $A$ or $B$, but not the sum, $A+B$; see [3, 13] and [4].

A classical splitting method for solving problem (1) is the so called Forward-Backward splitting method as proposed in [14]. Assuming that $domB \subseteq domA$, the scheme is given as follows :

$$x^{k+1} = (I + \beta_k B)^{-1}(I - \beta_k A)(x^k). \tag{2}$$

Where $\beta_k > 0$ for all $k$. The iteration defined by (2) converges when the inverse of the forward mapping is strongly monotone as well as over other undesired assumptions on the stepsize $\beta_k$ and the operator $B$; see, for instance, [14] and [15]. An important and promising modification of Scheme (2) was presented by Tseng in [13]. It consists in:

$$J(x^k, \beta_k) = (I + \beta_k B)^{-1}(I - \beta_k A)(x^k); \tag{3}$$

$$x^{k+1} = P_X(J(x^k, \beta_k) - \beta_k[A(J(x^k, \beta_k)) - A(x^k)]), \tag{4}$$

Where $X$ is a suitable nonempty, closed and convex set, belonging to $dom(A)$. The stepsize $\beta_k$ is chosen to be the largest $\beta \in \{\sigma, \sigma\theta, \sigma\theta 2...\}$ satisfying:

$$\beta\|A(J(x^k, \beta))\| \leq \delta\|J(x^k, \beta) - x^k\|, \tag{5}$$

With $\theta, \delta \in (0,1)$ and $\sigma>0$. Note that there exists various choices for the set $X$. If $dom(B)$ is closed, then the result of Minty in [9], implies that $dom(B)$ is convex, hence we may choose $X = dom(B)$; see [13].

The convergence of (3)−(5) was established assuming maximal monotonicity of $A$ and $B$, as well as Lipschitz continuity of $A$. It is important to say that, in the above scheme, in order to compute satisfying (5), the forward-backward operator (3) must be calculated, in each tentative of the step. From a computational point of view, this represents a considerable drawback.

In order to overcome these two serious limitations a algorithm has been proposed. We show the convergence to a solution of problem (1), assuming only monotonicity of both operators however without demands Lipschitz continuity of A. Our approach contains two parts. The first being a separating half space, containing the solution set of the problem, is found. This procedure employs a new Armijo-type search which performs only one calculation of the forward-backward operator instead Tseng's algorithm.

When $B=N_X$ , problem (1) may be written as $0 \in A(x) + N_X(x)$; (1) collapses to classic variational inequality problems VIP($X, A$), i.e. to find a vector $x \in X$ such that

$$\langle A(x), y - x\rangle \geq 0, \ \forall y \in X. \tag{6}$$

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This problem is the well studied variational inequality problem with numerous applicationsin optimization theory; see [7, 11]. An excellent survey of projection methods for variational inequality problems can be found in [5].

This work is organized as follows. The next section provides some preliminary results that will be used in the remainder of this paper. Section 3 introduces the algorithm and proves its convergence.

## 2. Preliminaries

First of all, we introduce the notation. Let $R^n$ be a n-dimensional Euclidean space and $X$ be a nonempty closed convex subset of $R^n$. $(\cdot,\cdot)$ denotes the usual inner product in $R^n$. The solution set is denoted by $S^* := \{x \in R^n : 0 \in (A+B)(x)\}$.

The distance from a point $x \in \mathrm{R}^n$ to $X$ denoted by dist($x$, $X$). The projection from a point

$x$ onto $X$ denoted by $P_X(x)$. Since $X$ is a nonempty closed and convex set, we have

$$\mathrm{dist}(x, P_X(x)) := \inf\{\|x-y\| \mid y \in X\}.$$

Recall that an operator $T$: $R^n \rightrightarrows R^n$ is monotone if, for all $(x, u), (y, v) \in GrT := \{(x, u) \in R^n \times R^n : u \in T(x)\}$, we have $(x-y, u-v) \geq 0$, and it is maximal if T has no proper monotone extension in the graph inclusion sense.

In this section, we present some definitions and results needed for the convergence analysis of the proposed methods. First, we state two well-know facts on orthogonal projections.

**Proposition 2.1.** Let $X \subset \mathrm{R}^n$ be a nonempty closed convex set. Then we have

(i) $z = P_X(x)$ if and only if $z \in X$ and $\langle z - x, y - z\rangle \geq 0,\ \forall y \in X$;

(ii) $\|P_X(x) - x\|^2 \leq \|x - z\|^2 - \|P_X(x) - z\|^2,\ \forall x \in \mathbb{R}^n, \forall z \in X$.

(iii) $P_X = J_{N_X} = (I + N_X)^{-1}$.

*Proof.* (i) and (ii) See Lemma 1.1 and 1.2 in [18]. (iii) See Proposition 2.3 in [1].

In the following we state some useful results on maximal monotone operators.

**Lemma 2.2.** Let $T$: $dom(T) \subseteq R^n \Rightarrow R^n$ be a maximal monotone operator. Then

(i) $Gr(T)$ is closed.

(ii) $T$ is bounded on bounded subsets of the interior of its domain.

*Proof.* (i) See Proposition 4.2.1 in [20].

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(ii) See Lemma 5 in [19].

**Proposition 2.3.** Let $T$: $domT \subseteq R^n \rightrightarrows R^n$ be a point-to-set and maximal monotone operator. Given $\beta > 0$ then the operator $(I + \beta T)^{-1}$: $R^n \to domT$ is single valued and maximal monotone.

*Proof.* See Theorem 4 in [10].

**Proposition 2.4.** Given $\beta > 0$ and $A$: $dom(A) \subseteq R^n \to R^n$ and $B$: $domB \subseteq R^n \rightrightarrows R^n$ two maximal monotone operators, then

$$x = (I + \beta B)^{-1}(I - \beta A)(x)$$

If and only if, $0 \in (A + B)(x)$

*Proof.* See Proposition 3.13 in [3].

Now, we define the so-called Fej´er convergence.

**Definition 2.5.** Let $S$ be a nonempty subset of $R^n$. A sequence $\{x^k\} \subset R^n$ is said to be convergent Fejér to $S$, if and only if, for all $x \in S$ there exists $k_0 \geq 0$, such that $\|x^{k+1} - x\| \leq \|x^k - x\|$ for all $k \geq k_0$.

This definition was introduced in [17] and has been elaborated further in [6] and [8]. A useful result on Fej´er sequences is the following.

**Proposition 2.6.** If $\{x^k\}$ is Fejér convergent to $S$. Then

(i) the sequence $\{x^k\}$ is bounded;

(ii) the sequence $\{\|x^k - x\|\}$ is convergent for all $x \in S$.

(iii) if a cluster point $x^*$ belongs to $S$, then the sequence $\{x^k\}$ converges to $x^*$.

*Proof.* (i) and (ii) See Proposition 5.4 in [21]. (iii) See Theorem 5.5 in [21].

## 3. Algorithm

Let $A$: $domA \subseteq R^n \to R^n$ and $B$: $domB \subseteq R^n \rightrightarrows R^n$ be two maximal monotone operators, with $A$ point-to-point and $B$ point-to-set. Assume that $domB \subseteq domA$. Choose any nonempty, closed and convex set, $X \subseteq domB$, satisfying $X \cap S^* \neq \emptyset$. Thus, from now on, the solution set, $S^*$ is nonempty. Also we assume that the operator $B$ satisfies, that for each bounded subset V of $domB$ there exists $R > 0$, such that $B(x) \cap \mathrm{B}[0, R] \neq \emptyset$, for all $x \in V$.

Let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence such that $\{\beta_k\} \subseteq [\check{\beta}, \hat{\beta}]$ with $0 < \check{\beta} \leq \hat{\beta} < \infty$.

**Algorithm 1.** Choose parameters $\delta$ and $\theta \in (0,1)$. Take $x_0 \in X$. Set $k = 0$.

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**Step 1.** Given $\beta_k$ and $x^k$, compute the forward-backward operator at $x^k$,

$$J(x^k, \beta_k) := (I + \beta_k B)^{-1}(I - \beta_k A)(x^k). \quad (7)$$

If $x^k = J(x^k, \beta_k)$, stop. Otherwise, go to Step 2.

**Step 2. (Armijo-type search)** Otherwise, begin the inner loop over $j$.

Put $j = 0$ and chose any $u_j^k \in B(\theta^j J(x^k, \beta_k) + (1-\theta^j)x^k) \cap \mathbb{B}[0, R]$. If

$$\langle A(\theta^j J(x^k, \beta_k) + (1-\theta^j)x^k) + u_j^k, x^k - J(x^k, \beta_k)\rangle \geq \frac{\delta}{\beta_k}\|x^k - J(x^k, \beta_k)\|^2 \quad (8)$$

Then $j(k) = j$ and stop. Else, $j = j + 1$.

**Step 3.** Compute $x^{k+1} := P_{H_k}(x_k)$,where $H_k := H(\bar{x}^k, \bar{u}^k)$ is a half-space and defined by the function

$$H(x, u) := \{y \in R^n : \langle A(x) + u, y - x\rangle \leq 0\} \quad (9)$$

$$\alpha_k := \theta^j(k), \quad (10)$$

$$\bar{u}^k = u_{j(k)}^k, \quad (11)$$

$$\bar{x}^k := \alpha_k J(x^k, \beta_k) + (1 - \alpha_k)x^k \quad (12)$$

**Step 4.** Let $k = k + 1$ and return to Step 1.

## 4. Convergence analysis

In this section we analyze the convergence of the algorithms presented in the previous section. First, we present some general properties as well as prove the well-definition of the algorithm.

**Lemma 4.1.** For all (x, u) ∈ Gr(B), $S_* \subseteq H(x, u)$.

*Proof.* Take $x^* \in S^*$. Using the definition of the solution, there exists $v^* \in B(x^*)$, such that $0 = A(x^*) + v^*$. By the monotonicity of $A + B$, we have

$$\langle A(x) + u - (A(x^*) + v^*), x - x^*\rangle \geq 0$$

for all $(x, u) \in Gr(B)$. Hence,

and by (9), $x_* \in H(x, u)$.

From now on, $\{x^k\}$ is the sequence generated by the algorithm.

**Proposition 4.2.** The algorithm is well-defined

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*Proof.* By Proposition 2.4, Stop Step 1 is well-defined. The proof of the well definition of $j(k)$ is by contradiction. Assume that for all $j \geq 0$ having chosen

$u_j^k \in B(\theta^j J(x^k, \beta_k) + (1-\theta^j)x^k) \cap B[0, R]$

$$\langle \beta_k A(x^k) + \beta_k u^k, x^k - J(x^k, \beta_k)\rangle \leq \delta \|x^k - J(x^k, \beta_k)\|^2 \tag{13}$$

It follows from (7) that

$$\beta_k A(x^k) = x^k - J(x^k, \beta_k) - \beta_k v^k$$

For some $v^k \in B(J(x^k, \beta_k))$.Now, the above equality together with (13) lead to

$$\|x^k - J(x^k, \beta_k)\|^2 \leq \langle x^k - J(x^k, \beta_k) - \beta_k v^k + \beta_k u^k, x^k - J(x^k, \beta_k)\rangle \leq \delta \|x^k - J(x^k, \beta_k)\|^2$$

Using the monotonicity of $B$ for the first inequality. So,

$$(1-\delta)\|x^k - J(x^k, \beta_k)\|^2 \leq 0$$

Which contradicts Step 1. Thus, the algorithm is well defined.

**Proposition 4.3.** $x^k \in H(\bar{x}^k, \bar{u}^k)$ for $\bar{x}^k$ and $\bar{u}^k$ as in (12) and (11), respectively, if and only if, $x^k \in S^*$.

*Proof.* Since $x^k \in H(\bar{x}^k, \bar{u}^k)$, $\langle A(\bar{x}^k) + \bar{u}^k, x^k - \bar{x}^k\rangle \leq 0$. Using the Armijo-type search, given in

(8) and (12), we obtain

$$0 \geq \langle A(\bar{x}^k) + \bar{u}^k, x^k - \bar{x}^k\rangle = \alpha_k \langle A(\bar{x}^k) + \bar{u}^k, x^k - J(x^k, \beta_k)\rangle \geq \frac{\alpha_k}{\beta_k}\delta \|x^k - J(x^k, \beta_k)\|^2 \geq 0$$

which implies that $x^k = J(x^k, \beta_k)$. So, by Proposition 2.4, $x^k \in S^*$. Conversely, if $x^k \in S^*$, using Lemma 4.1, $x^k \in H(\bar{x}^k, \bar{u}^k)$.

**Lemma 4.4.** Let $x^k$ and $\{\alpha_k\}$ be sequences generated by the algorithm. With $\delta$ and $\hat{\beta}$ as in the algorithm.

$$\langle A(\bar{x}^k) + \bar{u}^k, x^k - \bar{x}^k\rangle \geq \frac{\alpha^k \delta}{\hat{\beta}} \|x^k - J(x^k, \beta_k)\|^2 \geq 0, \tag{14}$$

for all $k$.

**Proposition 4.5.** If Algorithm stops, then $x^k \in S^*$

*Proof.* If Step 3 is satisfied, $x^{k+1} = P_H(x^k) = x^k$. implying that $x^k \in H_k$ and by Proposition 4.3, $x^k \in S^*$.

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From now on assume that Algorithm does not stop. Note that by Lemma $H_k$ is nonempty for all $k$. Then the projection step is well defined, i.e. if Algorithm does not stop, it generates an infinite sequence $\{x^k\}$.

**Proposition 4.6.**

(i) the sequence $\{x^k\}$ is Fejér convergent to $S^* \cap X$.

(ii) the sequence $\{x^k\}$ is bounded.

(iii) $\lim_{k\to\infty}\langle A(\bar{x}^k) + \bar{u}^k, x^k - \bar{x}^k\rangle = 0$. *Proof.* (i) Take $x^* \in S^* \cap X$.

Note that, by definition $(\bar{x}^k, \bar{u}^k) \in Gr(B)$. Using Proposition 2.1 and Lemma 4.1, we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|P_{H_k}(x^k) - P_{H_k}(x^*)\|^2 \\ &\le \|x^k - x^*\|^2 - \|P_{H_k}(x^k) - x^k\|^2. \end{aligned} \tag{15}$$

SO, $\|x^{k+1} - x^*\| \le \|x^k - x^*\|$.

(ii) Follows immediately from the previous item.

(iii) Take $x^* \in S^* \cap X$. Using (11) and

$$P_{H_k}(x^k) = x^k - \frac{\langle A(\bar{x}^k) + \bar{u}^k, x^k - \bar{x}^k\rangle}{\|A(\bar{x}^k) + \bar{u}^k\|^2}(A(\bar{x}^k) + \bar{u}^k), \tag{16}$$

Combining with (15), yields

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\le \|x^k - x^*\|^2 - \|x^k - \frac{\langle A(\bar{x}^k) + \bar{u}^k, x^k - \bar{x}^k\rangle}{\|A(\bar{x}^k) + \bar{u}^k\|^2}(A(\bar{x}^k) + \bar{u}^k) - x^k\|^2 \\ &= \|x^k - x^*\|^2 - \frac{(\langle A(\bar{x}^k) + \bar{u}^k, x^k - \bar{x}^k\rangle)^2}{\|A(\bar{x}^k) + \bar{u}^k\|^2}. \end{aligned}$$

Reordering the above inequality, we get

$$\frac{(\langle A(\bar{x}^k) + \bar{u}^k, x^k - \bar{x}^k\rangle)^2}{\|A(\bar{x}^k) + \bar{u}^k\|^2} \le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2. \tag{17}$$

By Proposition 2.3 and the continuity of $A$, we have that J is continuous. Since $\{x^k\}$ and $\{\beta_k\}$ are bounded then $\{J(x^k, \beta_k)\}$ and $\{\bar{x}^k\}$ are bounded, implying the boundedness of $\{A(\bar{x}^k)+\bar{u}^k\}$. Using Proposition 2.6, the right side of (17) goes to 0, when k goes to $\infty$, establishing the result.

Next, we establish our main convergence result on Algorithm.

**Theorem 4.7.** The sequence $\{x^k\}$ converges to some element belonging to $S^* \cap X$.

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*Proof.* We claim that there exists a cluster point of $\{x^k\}$ belonging to $S^*$. The existence of the cluster points follows from Proposition. Let $\{x^{ik}\}$ be a convergent subsequence of $\{x^k\}$ such that, $\{\bar{x}^{i_k}\}, \{\bar{u}^{i_k}\}, \{\alpha_{i_k}\}$

are convergent, and set $\lim_{k\to\infty} x^{i_k} = \tilde{x}$. Using Proposition 4.6 and taking limits in (14) over the subsequence $\{i^k\}$, we have

$$0 = \lim_{k\to\infty} \langle A(\bar{x}^{i_k}) + \bar{u}^{i_k}, x^{i_k} - \bar{x}^{i_k}\rangle \geq \lim_{k\to\infty} \frac{\alpha_{i_k}\delta}{\hat{\beta}} \|x^{i_k} - J(x^{i_k}, \beta_{i_k})\|^2 \geq 0. \tag{18}$$

Therefore,

$$\lim_{k\to\infty} \alpha_{i_k} \|x^{i_k} - J(x^{i_k}, \beta_{i_k})\| = 0$$

Now, consider the two possible cases.

Let $\bar{\alpha} := \inf_{j\in I}\{\alpha_{i_{k_j}}\}$. Two cases are be considered.

Case A: If $\bar{\alpha} > 0$, then $\alpha_{i_{k_j}} > \bar{\alpha}$ for all $j \in I$. We deduce that

$$\lim_{j\to\infty} \|x^{i_{k_j}} - J(x^{i_{k_j}} - \beta_{i_{k_j}})\| = 0. \tag{19}$$

Taking a subsequence, if necessary, we may assume that $\lim_{k\to\infty} \beta_{i_k} = \tilde{\beta}$. Such that $\tilde{\beta} \geq \breve{\beta} > 0$ and since J is continuous, by the continuity of A and $(I + \beta_k B)^{-1}$ and by Proposition 2.3, (19) becomes $\tilde{x} = J(\tilde{x}, \tilde{\beta})$,

which implies that $\tilde{x} \in S^*$. Establishing the claim.

Case B: If $\bar{\alpha} = 0$ , then there exists an index set $\bar{I} \subset I$ such that

$$\lim_{j\to\infty, j\in\bar{I}} \alpha_{i_{k_j}} = 0$$

We have that, for $\theta \in (0, 1)$, defined in the algorithm

$$\lim_{j\to\infty} \frac{\alpha_{i_{k_j}}}{\theta} = 0$$

Define

$$y^{i_{k_j}} := \frac{\alpha_{i_{k_j}}}{\theta} J(x^{i_{k_j}}, \beta_{i_{k_j}}) + (1 - \frac{\alpha_{i_{k_j}}}{\theta}) x^{i_{k_j}}$$

Then,

$$\lim_{j\to\infty} y^{i_{k_j}} = \tilde{x} \tag{20}$$

Using the definition of $j(k)$ and (10), $y^{i_{k_j}}$ does not satisfy (8), implying

$$\langle A(y^{i_{k_j}}) + u^{i_{k_j}}_{j(i_{k_j})-1}, x^{i_{k_j}} - J(x^{i_{k_j}}, \beta_{i_{k_j}})\rangle < \frac{\delta}{\beta_{i_{k_j}}} \|x^{i_{k_j}} - J(x^{i_{k_j}}, \beta_{i_{k_j}})\|^2 \tag{21}$$

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For $u^{i_{k_j}}_{j(i_{k_j})-1} \in B(y^{i_{k_j}})$ and for all $j \in \bar{I}$.

Redefining the subsequence $\{i_{k_j}\}$ , if necessary, we may assume that $\{\beta_{i_{k_j}}\}$ converges to some $\tilde{\beta}$ such that $\tilde{\beta} \geq \breve{\beta} > 0$ and $\{u^{i_{k_j}}_{j(i_{k_j})-1}\}_{j=0}^{\infty}$ converges to $\tilde{u}$ . By the maximality of *B*, $\tilde{u}$ belongs to $B(\tilde{x})$ . Using the continuity of *J*, $\{J(x^{i_{k_j}}, \beta_{i_{k_j}})\}$ converges to $J(\tilde{x}, \tilde{\beta})$. Using (20) and (21) taking limit in over the subsequence $\{i_{k_j}\}$, we have

$$\langle A(\tilde{x}) + \tilde{u}, \tilde{x} - J(\tilde{x}, \tilde{\beta})\rangle \leq \frac{\delta}{\tilde{\beta}}\|\tilde{x} - J(\tilde{x}, \tilde{\beta})\|^2. \tag{22}$$

Using (7) and multiplying by $\tilde{\beta}$ on both sides of (22) we get

$$\langle \tilde{x} - J(\tilde{x}, \tilde{\beta}) - \tilde{\beta}\tilde{v} + \tilde{\beta}\tilde{u}, \tilde{x} - J(\tilde{x}, \tilde{\beta})\rangle \leq \delta\|\tilde{x} - J(\tilde{x}, \tilde{\beta})\|^2.$$

Where $\tilde{v} \in B(J(\tilde{x}, \tilde{\beta}))$. Applying the monotonicity of B, we obtain

$$\|\tilde{x} - J(\tilde{x}, \tilde{\beta})\|^2 \leq \delta\|\tilde{x} - J(\tilde{x}, \tilde{\beta})\|^2,$$

implying that $\|\tilde{x} - J(\tilde{x}, \tilde{\beta})\| \leq 0$ . Thus, $\tilde{x} = J(\tilde{x}, \tilde{\beta})$ and hence, $\tilde{x} \in S*$.

This completes the proof.

## 5. Conclusion

In this paper, we have presented and studied a half-space projection method for finding a zero of the sum of two operators. Comparing with the known methods, this method can reduce one projection onto the strategy set per iteration. The global convergence is proved under the assumption of S∗ I= ∅. In fact, to sure this assumption is difficult. So, how to replace this assumption or how to weaken it could be the subject of a future research.

## References

[1] Bauschke, H. H., Burke, J. V., Deutsch, F. R., Hundal, H. S.: A new proximal point iter- ation that converges weakly but not in norm. Proceedings of the American Mathematical Society.133, 1829-1835 (2005)

[2] Bello Cruz, J.Y., Iusem, A.N.: A strongly convergent direct method for monotone varia- tional inequalities in Hilbert spaces. Numerical Functional Analysis and Optimization.30, 23-36 (2009)

[3] Eckstein, J.Splitting Methods for Monotone Operators, with Applications to Parallel Op- timization. PhD thesis, Massachusetts Institute of Techonology, Cambridge, MA, 1989. Report LIDS-TH-1877, Laboratory for Information and Decision Systems, M.I.T.

ISSN 2616-5775 Vol. 2, Issue 1: 92-101, DOI: 10.25236/AJCIS.010022

[4] Eckstein, J., Svaiter, B.F.: A family of projective splitting methods for the sum of two maximal monotone operators. Mathematical Programming. 111, 173-199 (2008)
[5] Facchinei, F., Pang, J.S.: Finite-dimensional Variational Inequalities and Complementar- ity Problems. Springer, Berlin. (2003)
[6] Bauschke, H.H.,Borwein, J.M. : On projection algorithms for solving convex feasibility problems. SIAM Review. 38, 367-426 (1996)
[7] Hartman, P., Stampacchia, G.: On some non-linear elliptic dierential-functional equations. Acta Ma-thematica. 115, 271-310 (1966)
[8] Iusem, A.N., Svaiter, B.F.: Entropy-like proximal methods in convex programming. Math- ematics of Operations Research .19,790-814 (1994).
[9] Minty, G.: On the maximal domain of a monotone function. Michigan Mathematical Journal. 8, 135-137 (1961)
[10] Minty, G.: Monotone (nonlinear) operators in Hilbert Space. Duke Mathetematical Journal. 29, 341-346 (1962)
[11] Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York. (1980)
[12] Solodov, M.V., Svaiter, B.F.: A new projection method for variational inequality problems. SIAM Journal on Control and Optimization. 37, 765-776 (1999)
[13] Tseng, P.: A modified forward-backward splitting method for maximal monotone map-pings. SIAM on Journal Control Optimization. 38, 431-446 (2000).
[14] Lions, P.L., Mercier, B. : Splitting algorithms for the sum of two nonlinear operators. SIAM of Journal Numerical Analysis. 16, 964-979 (1979)
[15] Passty, G.B.: Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. Journal of the Mathematical Analysis and Applications. 72, 383–390 (1979).
[16] Li, F.L., He, Y.R.: An algorithm for generalized variational inequality with pseudomono- tone mapping. J. Comput. Appl. Math. 228, 212–218 (2009)
[17] Browder, F.E.: Convergence theorems for sequences of nonlinear operators in Banach spaces. Mathema-tische Zeitschrift.100, 201–225 (1967)
[18] Zarantonello, E.H.: Projections on Convex Sets in Hilbert Space and Spectral Theory, Contributions to Nonlinear Functional Analysis. Academic Press, New York, NY, (1971)
[19] Bello Cruz, J.Y., Iusem, A.N.: Convergence of direct methods for paramonotone variational inequalities. Computation Optimization and Applications. 46, 247–263 (2010)
[20] Burachik, R.S., Iusem, A.N.: Set-Valued Mappings and Enlargements of Monotone Oper- ators. Springer, Berlin (2008)
[21] Bauschke HH.,Combettes PL.: Convex analysis and monotone operator theory in Hilbert spaces. New York (NY):Springer. (2011).