ISSN 2522-6398 Vol. 6, Issue 13: 109-113, DOI: 10.25236/FER.2023.061318

Application of Proof by Contradiction in Solving Junior High School Mathematics Competition Questions

Fenglin Wu[a], Zhiwang Liu[b,*]

School of Mathematics and Statistics, Ningxia University, Yinchuan, 750000, China
[a]w18083028232@163.com, [b]1415483779@qq.com
**Corresponding author*

Abstract: *Mathematics competition is an important platform to improve students' thinking and ability, so the questions of mathematics competition are more difficult and more comprehensive than ordinary questions. The examination of proof by contradiction in junior high school mathematics competitions can improve students' thinking and ability in many aspects and has an educational value that cannot be ignored. The author gives examples to explore the specific application of proof by contradiction in solving junior high school mathematics competition questions. Analyze its investigation value in the competition.*

Keywords: *proof by contradiction; A math contest; Educational value*

1. A brief introduction to paradoxical proofs

1.1 The concept of proof by contradiction

Proof by contradiction is an indirect method of solving problems, which starts from the idea contrary to the original conclusion, and proves that the conclusion contrary to the original conclusion is contrary to the known conditions, hypotheses, or known axioms, theorems, formulas, properties, etc., to get the correctness of the original conclusion. Especially in the face of some difficult mathematical problems, it may be easier to think about the problem from the opposite perspective, so as toify the complex problem.

There is a short story that illustrates the paradox well. Three ancient Greek philosophers were sitting under a tree arguing about philosophy. They were so tired that they fell asleep against the tree. Then a mischievous passer-by blackened all three of their foreheads with black charcoal. When the three men woke up, they looked at each other's foreheads and smiled. Everyone didn't realize their foreheads were blackened because everyone thought the other two were making fun of each other. One of them suddenly stopped laughing, for he noticed that his forehead had been blackened, too. How did this person realize that his forehead was also blackened? Can you explain?

Answer: Set A, B, and C represent three philosophers, may as well set A's face is not blackened. So A thinks: "None of us can see our forehead, so all three of us think our faces are not blackened, and if my face is really not blackened, then B and C can see it, and B should laugh at C's surprise since B sees my face is not blackened, and he thinks his face is not blackened." Because B and I are not blackened, C also thinks that his face is not blackened ed, and C has no reason to laugh. However, now C is laughing, and B is not surprised by C's smile, so B thinks C is laughing at me. This contradicts the assumption that my forehead is not blackened. As you can see, my face is also blacked out. It should be emphasized here that A did not directly see whether his face was blacked out, he analyzed and thought based on the expressions of B and C, and explained that his face was blacked out. Simply put, A is aware of its face being blackened by stating that the opposite of the face being blackened is wrong not to be blackened.

Thus, to demonstrate a conclusion is true, we don’t prove it directly from the front, but by showing that its opposite is false, and thus that it is itself correct, which is called "proof by contradiction."

1.2 The applicable question type of proof by contradiction

In the process of solving competition problems, if we directly prove difficult problems, we must think of using the method of proof by contradiction, the author systematically gives five kinds of questions to

ISSN 2522-6398 Vol. 6, Issue 13: 109-113, DOI: 10.25236/FER.2023.061318

understand the method of proof by contradiction.

"Negative" propositions: propositions with words such as "not", "not belonging", or "not having certain sexual qualities", such as"ΔABC is not a right triangle", "$\sqrt{5}$ does not belong to rational numbers", "$\angle A$ is not complementary to $\angle B$" and so on.

"Finiteness" and "infinity" propositions: propositions have "finite", "infinite" and other words, such as "the number that meets the conditions is finite", "there are infinite elements" and so on.

"Existence" proposition: The proposition has "existence", "there" and other words, such as "there is a right triangle", "there are even numbers in these numbers" and so on.

"At least" and "at most" propositions: propositions have words such as "at least" and "at most", such as "at least one solution" and "at most two points of intersection".

A proposition with too few known conditions: A proposition with too few known conditions is difficult to prove, and the conclusion contrary to the original conclusion is contrary to the known conditions, hypotheses, or known axioms, theorems, formulas, properties, etc [1].

1.3 The steps of solving the problem by proof by contradiction

The steps of solving the problem of the method of proof by contradiction should go through three steps "anti-hypothesis - return to error - conclusion".

Negative conclusion (negate): A proposition that assumes the opposite of the proven conclusion exists.

Deduced contradictions (fallacies): The deduced contradictions include contradictions with known conditions, contradictions with assumptions, or contradictions with axioms, theorems, formulas, properties, etc.

Positive conclusion (conclusion): Based on the deduced contradiction, it can be shown that the proposed hypothesis is not valid, thus affirming that the original proposition is valid [2].

2. Overall and local disproofs in junior middle school mathematics competition

Let's take a look at the following two examples of junior high school math competitions.

Example 1 (Suzhou Junior High School Mathematics Competition in 1985) to verify: 1986 cannot be equal to the value of any quadratic equation with integer coefficients of $ax^2+bx+c=0$.

Analyze Suppose there exists a quadratic equation with integer coefficients of $ax^2+bx+c=0$ whose discriminant is equal to 1986.

Suppose $\Delta= b^2-4ac=1986=4k+2$, Then b^2 is divisible by 2, so b is an even number, which is $b=2t$, then $b^2=4t^2$ and $4t^2-4ac=4k+2$.In this case, the number on the left side of the equation is divisible by 4, and the number on the right side cannot be integer by 4. So the hypothesis is not true. The original statement is true.

Example 2(2019 National Junior High School Mathematics Joint Competition Test) Let's say p is a prime number greater than 2, and k is a positive integer, If the horizontal coordinate of the graph of $y=x^2+px+(k+1)p-4$ the function of and the two points of the x-axis has at least one integer, find the value of k.

Analyze The two roots of the equation of $y=x^2+px+(k+1)p-4$are x_1 and x_2, at least one of them is an integer, obtained $x_1+x_2=-p$, $x_1x_2=(k+1)p-4$ by the relationship between the roots and the coefficients, and thus have

$$(x_1+2)(x_2+2)=x_1x_2+2(x_1+x_2)+4=(k-1)p \quad (1)$$

If $k=1$, then the equation is $x^2+px+2(p-2)=0$, it has the sum of two integer roots of -2 and $2-p$.

If $k>1$, then $k-1>0$.

Because t+$x_2=-p$ is an integer, if at least one of x_1 and x_2 is an integer, and x_2 are all integers. And because p is a prime number, $p|x_1+2$ or $p|x_2+2$ are known by the formula (1).

If you wish to set up $p|x_1 + 2$, you can set up $x_1 + 2 = mp$(where m is a non-zero integer),$x_2 + 2 = \frac{k-1}{m}$ Can be obtained by formula (1), so $(x_1 + 2) + (x_2 + 2) = mp + \frac{k-1}{m}$.

In that way, $x_1 + x_2 + 4 = mp + \frac{k-1}{m}$.

Also because $x_1 + x_2 = -p$, so $-p + 4 = mp + \frac{k-1}{m}$,

In that way

$$(m+1)p + \frac{k-1}{m} = 4 \tag{2}$$

If m is a positive integer, then $(m+1)p \geq (1+1) \times 3 = 6$, $\frac{k-1}{m} > 0$, so $(m+1)p + \frac{k-1}{m} > 6$, this is in contradiction with the (2) formula.

If m is a negative integer, then $(m+1)p < 0$, $\frac{k-1}{m} < 0$, so $(m+1)p + \frac{k-1}{m} < 0$, this is in contradiction with the (2) formula. To sum up, when $k > 1$, the equation of $x^2 + px + (k+1)p - 4 = 0$ can't have integer roots.

To sum up, $k = 1$.

In the above example, example 1 is the global disproof, and Example 2 is the local disproof.

In the proof of Example 1, we begin by assuming that the negative statement of the original statement is true and then prove to find evidence that contradicts this hypothesis. The word "cannot be equal" appears in the question, which belongs to the negative life question.

In the proof of example 2, it belongs to the classification discussion method as a whole, because k is a positive integer, So let's start by discussing whether $k = 1$ whether is Conform to the meaning of the question, and then discuss Whether the situation of k>1 is consistent with the meaning of the question, but from a local point of view, in the discussion the case of k>1, the inverse proof method is used. When assuming m is positive and negative integers, it is found that both cases are contradictory with the obtained formula, so obtaining the case of k>1 does not meet the requirements. Therefore, this is a partial disproof in the categorical discussion. The word "at least" appears in the question, which belongs to the class of propositions of "at least" and "at most". In addition, local disproof is also used in other proofs, such as construction method, induction method, etc. Local disproof is widely used in reasoning proofs as a whole.

3. The use of paradoxical proof to prove the judgment-type proposition and the exploration-type proposition

Example 3 Proof:$\sqrt{2}$ is not a rational number.

Analyze Assuming $\sqrt{2}$ is a rational number, Then a, b exists to make $\sqrt{2} = \frac{b}{a}$(where a, b are positive natural numbers and mutual prime).

$\therefore 2 = \frac{b^2}{a^2}$, so $b^2 = 2a^2$.

$\therefore$ b^2 is divisible by 2,$\therefore$ and b is also divisible by 2.

If $b = 2p$(p is a positive natural number), so $2a^2 = b^2 = 4p^2$, $\therefore a^2 = 2p^2$.

So a^2 is divisible by 2, and a is also divisible by 2.

$\therefore$a and b have common factors of 2, which contradicts the primacy of a and b.

$\therefore$The hypothesis that $\sqrt{2}$ is rational is not valid,$\sqrt{2}$ is not rational.

$\therefore$If the hypothesis is not true, the original proposition is true.

Example 4 Given that a and b are positive integers, find the smallest positive integer value $M = 3a^2 - ab^2 - 2b - 4$ can get.

Analyze Because “a” and “b” are positive integers, to make the value of $M = 3a^2 - ab^2 - 2b - 4$ a positive integer, there is $a \geq 2$.

ISSN 2522-6398 Vol. 6, Issue 13: 109-113, DOI: 10.25236/FER.2023.061318

When $a = 2$,b can only be 1, now $M = 4$. Therefore, the minimum positive integer value M can be obtained is not more than 4.

When $a = 3$,b can only be 1 or 2.If $b = 1$, $M = 18$; If $b = 2$, $M = 7$.

When $a = 4$,b can only be 1 or 2 or 3.If $b = 1$, $M = 38$; If $b = 2$, $M = 24$; If $b = 3$, $M = 2$.

(Consider this: Can $M = 3a^2 - ab^2 - 2b - 4$ be equal to 1?)

(proof by contradiction) hypothesis $M = 1, 3a^2 - ab^2 - 2b - 4 = 1$,

$$\text{namely } 3a^2 - ab^2 = 2b + 5, a(3a - b^2) = 2b + 5 \tag{3}$$

Because b is a positive integer, $2b + 5$ is odd, so a is odd, b is even,

May as well set $a = 2m + 1, b = 2n$, Where m and n are both positive integers, then

$$a(3a - b^2) = (2m + 1)[3(2m + 1) - (2n)^2] = 4(3m^2 + 3m - 2mn^2 - n^2) + 3.$$

Namely, The remainder of $a(3a - b^2)$ divided by 4 is 3, and the remainder of $2b + 5 = 2(2n) + 1 = 4n + 1)$ divided by 4 is 1, so the formula (3) is impossible, so $M \neq 1$. Therefore, the smallest positive integer value that M can obtain is 2.

In the above example, example 3 belongs to the use of paradoxical proof to solve the judgment statement, which is the proposition that directly gives the conclusion, and belongs to the proof problem. The word "not" appears in the question, which belongs to the "negative" proposition.

Example 4 belongs to the application of paradoxical proof to solve the exploratory proposition. The exploratory proposition does not give a direct conclusion. It is not a proof problem, but a solution problem that requires the questioner to explore the conclusion. This topic belongs to the proposition of "too few known conditions".

4. The significance of proof by contradiction is examined in the competition

Junior high school students are in the stage of rapid development of thinking and ability. The significance of mathematical competition is not only to train students' problem-solving skills but also to promote students' thinking and ability improvement in the process of preparing for the competition and to get new perspectives and ways to solve problems.

4.1 Proof by contradiction promotes the development of students' core quality

The cultivation of core quality is the direction of the new curriculum reform in primary and secondary schools, and the mathematics competition also attaches great importance to the cultivation of students' core quality during the investigation process.

4.1.1 The method of inverse proof promotes the improvement of students' mathematical operation literacy

The amount of arithmetic in junior high school has increased significantly compared with that in primary school. Arithmetic exists in almost every problem type and solution in junior high school mathematics, and arithmetic is indispensable when solving problems by contradiction. Compared with the ordinary exam, the calculation in the competition questions is more complex and ingenious, which is very beneficial to the improvement of students' mathematical operation literacy.

4.1.2 Proof by contradiction promotes students' logical reasoning accomplishment

When using proof by contradiction to solve a competition problem, students should first analyze the known conditions, clarify the logic of knowledge, make assumptions, think about contradictions, and come up with solutions.

4.2 Proof by contradiction can improve students' other mathematical thinking ability

4.2.1 Promote students' innovative thinking

The questions of the math competition are innovative, many of which are combined with mathematical culture and realistic situations, and students will be influenced by innovative thinking in

ISSN 2522-6398 Vol. 6, Issue 13: 109-113, DOI: 10.25236/FER.2023.061318

this process. The solution to the math competition problem is also innovative, and there may be multiple solutions to a problem, encouraging students to use more ingenious solutions.

4.2.2 Enhance Students' reverse thinking

Different from other types of solutions, inverse proof starts from the opposite side of the original problem, which is a kind of reverse thinking. Students will improve their reverse thinking in the process of solving problems, and reverse thinking is an essential way to solve problems and practical problems in the future.

4.2.3 Cultivate students' rigorous thinking

Mathematics is a rigorous science, and if it's not rigorous, the whole proof can be overturned. The proof by contradiction itself is a rigorous method of proof, and the law of exclusion and contradiction is its basic laws of logic, which cannot be overturned. To solve the competition problem with the reverse solution, we must first assume the opposite of the original proposition, carefully list all the circumstances that contradict the hypothesis, and then negate it, without missing anything [3].

4.3 Improve problem-solving efficiency

The method of inverse proof is a kind of problem-solving method summarized by predecessors, which can be directly applied according to the meaning of the problem and through the fixed steps in the process of solving the problem. In the competition, when it is difficult to use direct proof, paradoxical proof can make people learn and solve problems, save valuable exam time, and greatly improve the efficiency of solving problems.

5. Conclusion

Proof by contradiction is a commonly used method to solve junior high school competition problems. The steps are almost fixed when using proof by contradiction, the key is to find contradictions. In junior high school mathematics competitions, proof by contradiction includes overall proof and local proof, and question types include judgment type and exploration type. The application of proof by contradiction in junior middle school mathematics competition can not only improve students' problem-solving efficiency but also have a positive value to the cultivation of students' core quality and the improvement of their thinking ability.

References

[1] Liu Fushan. Proof by Contradiction in Middle School Mathematics Competition [J]. Communication on Mathematics Teaching, 2015.
[2] Zhang Meng. Discussion on Proof by Contradiction in Middle School Mathematics [J]. Scientific Advice (Education and Research), 2019.
[3] Wang Fengchun. Strengthening the Teaching of Proof by Contradiction to Cultivate the Ability of Logical Reasoning [J]. Shanghai Middle School Mathematics, 2021.