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The Application of Calculus to Gauss's Theorem for Electrostatic Fields

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Abstract: *Lots of advanced mathematical knowledge has been applied in electromagnetism which is definitely one of the vital basic courses for physics, especially when the abstract calculus relations are used to express incomprehensible theorems and laws, the difficulty of learning is apparently increased. For those non-physics specialties but science-technology students, earning the electromagnetism in college also becomes a great challenge. This paper takes Gauss's theorem of electrostatic fields in electromagnetism as an example to explain the application of calculus in Gaussian theorem of electrostatic fields, which aims to, on the basis of conciseness and acceptability, help learners master the related knowledge and brings inspiration to the future teaching process.*

Keywords: *Calculus, Electromagnetism, Electrostatic fields, Gauss's theorem*

1. Introduction

Calculus, as one great achievement of mankind in the field of mathematics but also an extremely vital branch of Advanced Mathematics, not only allows humankind to recognize a problem flexibly but also provides convenience for the study and application of variables in functions. “In mathematical analysis, when the segmentation process is carried on indefinitely, local scope becomes infinitely small, which is called differentiation. And the sum of all the infinitely differential elements is integration”[1]. Those thoughts and methods of calculus reflect the law of unity of opposites between part and whole, finite and infinite, complexity and simplification, variate and constant, and approximation and accuracy. “Infinite” and “variation” indeed run through the application range of integral. A quantity, changing with space and time, is considered to be invariant as the amount of variability is small in tiny time-pace domain. At this instant or in a small region, applying the relevant laws to determine a functional relation between one quantity and other quantities is an integrand expression .Then, the process of adding infinite quantities determined by the same laws within a certain range of variation is integration. Therefore, the integral, which uses dialectical thinking to cope with contradictions between “finite” and “infinite”, “change” and “unchanged”, realizes the unity of opposites.

In physics, finite partitions for the scales of changed time and space are always carried out, among which approximate treatment is conducted; each finite range will become infinitely small, if the partitions are proceeded infinitely, and the approximations will gradually converge to accurate results. In physics courses, it is quite universal to utilize calculus to solve physical problems such as particle kinematics and dynamics, and electrostatic fields and steady magnetic fields, etc. Physical phenomena and laws are always based on the simplest phenomena and laws. In practice, complicated physical problems can be broken up into parts and divided into numerous ones in a tiny local range. And when divided into parts enough small, the physical problem can be regarded as the simplest and most basic one. Then, by accumulating and summing the results of problems in local ranges, the complex physical problems will be figured out.

The thoughts and methods of calculus are the basic ones for learning and solving physics problems. Electromagnetism, one representative example, applies a large number of thoughts of differentiation and integral to interpret relevant laws. Aren’t there any connections between Gauss's Theorem for Electrostatic Fields in Electromagnetism and Gauss's Theorem in Advanced Mathematics? Gauss's Theorem for Electrostatic Fields in Electromagnetism can be expressed by integral form as well as differential form, both of which can be transformed into each other. What perspectives should be held to understand the transformation process? Thus, this article aims to expound how to represent the Gauss theorem of electrostatic field by using calculus methods and the mutual transformation and the significance between integral form and differential form. Based on the content, the significance and necessity of calculus for the course of "Electromagnetics" are further illustrated. Meanwhile, on the basis of conciseness and acceptability, it further clarifies how the application of calculus in Gauss's theorem inspires teaching in practice and promotes learners' deep understanding and learning in this part of knowledge from the perspective of calculus thoughts, which will serve as a reference for the teaching process in the future.

2. The Traceability of Calculus in the History of Physics

Calculus is a critical branch in higher mathematics. "Functions are the basic objects of calculus study, limits are its fundamental concepts, and differentiation and integration are the limits of particular processes and particular forms" [2].

2.1. Traceability of Calculus Ideas in Ancient China

The idea of infinite divisibility and limit was put forward in the Taoist doctrine in the 7th century BC. There are many records in the development of China's geomancy, among which the concepts of finiteness, infiniteness and limits are clearly recorded in the Mojing. During the Spring-autumn Warring period, Huishi said, "A foot of flogging taken half of it in the day will not be exhausted for ten thousand generations." In the Three Kingdoms period, Liu Wei proposed "cyclotomic method" in the Nine Chapters on Mathematical Procedures - The thoughts "cutting the fine results in the loss of less, and cut and cut to the point that can not be cut, then integrating with the circumference of the body will lead to nothing to lose! "also embodies the ideas and methods of limits and infinitesimals. From Liu Wei's research on the volume of cones, circular platforms and cylinders to the process of Zu Heng's solution to the volume of a ball in the 5th century AD, those thoughts are reflected. In the Northern Song Dynasty, The "gap product technique" and "meeting circle technique", etc., proposed in Shen Kuo's "Mengxi Bitan", pioneered the study of summation of arithmetic series of higher[3-4].

2.2. Traceability of Calculus Ideas Abroad

The emergence of calculus thoughts in the West can be traced back to an ancient Greek scientist, Thales, whose studies focused on the area, volume and length of the ball and other issues before 7th century BC. "Method of exhaustion" in Elements of Geometry written by Euclid involves the integral method of limit thought. In ancient Greece, Archimedes' method of finding the area of a parabola also reflected the idea of summation of infinite series.

In modern times, based on the research outputs of predecessors, calculus, which was developed by Newton and Leibniz respectively, was established with the intuition of infinite small quantities as the starting point. It is generally believed that Newton focused on the study of calculus from a kinematic point of view, and invented the "positive flow of counting" (differentiation) and the "anti-streaming of counting (differential method) and "antifluid calculus" (integral method). His book: A Brief Treatise on Fluid Numbers was the first major document to elaborate calculus. Leibniz, a contemporary of Newton, studied calculus from a geometrical point of view. He realized that the tangent to a curve is based on the difference between the vertical and horizontal coordinates, and the quadrature is based on the sum of the areas of infinitely thin rectangles. Therefore, the integral as a summation operation is the inverse of the differential as a difference operation. Leibniz invented many symbols for calculus operations, which are still widely used today, such as dx, dy, ∫, and so on. According to the mentioned arguments, it is obvious that physics was a very significant factor in the creation and development of calculus. In the early days, calculus lacked a rigorous theoretical foundation but was easier to be understood in practice. Later, through the efforts of Weierstrass and Cauchy and others, calculus was built on the basis of limit theory. Then, the establishment of the theory of real numbers laid the theoretical foundation for the theory of limits. Thus, the basic theory of calculus and the method of thinking gradually became more and more perfect.

3. The Traceability of Calculus in the History of Physics

As we all know, the German Gauss (1777~1855) [5], one of the greatest scientists in modern times, is noted for his genius mathematical talent. His main achievements are the Binomial Theorem, the principle

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of "Least Squares", Gauss's Theorem on Vector Analysis, and proofs of the Fundamental Theorem of Algebra, etc., Gauss's Theorem in higher mathematics. Gauss's Theorem in Higher Mathematics included. However, it is not well known that Gauss was also highly accomplished in the field of physics, especially in electromagnetism. Gauss's Theorem in Electrostatic Fields was discussed when "The Universal Theory of Attractive and Repulsive Forces inversely proportional to the Square of the Distance" was completed in 1839. By analyzing Coulomb's law in vacuum, Gauss introduced Gauss's theorem in electrostatic field, which resulted in the name "Gauss's theorem".

Gauss's theorem in higher mathematics is expressed as follows: In a vector field $\vec{A}$(x,y,z), taking any closed surface S, which encloses a volume V, we have:

$$\oiint_S \vec{A} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{A} dV \tag{1}$$

The above equation shows that the flux of a vector field through any closed surface is equal to the integral of the dispersion in the volume it encloses. Using this law allows for the interconversion of the area component of a given closed surface and the volume component enclosed by that closed surface. At the same time, it provides a theoretical basis for the interconversion of the integral and differential expressions of Gauss's theorem for electrostatic fields later.

3.1. Application of Gauss's Theorem to Integration and Electric Flux in Electrostatic Fields

Electric flux is a physical quantity used to describe the nature of an electrostatic field. Here the electric flux is denoted by Ψ_e. Suppose a surface is taken in an electrostatic field, as in Figure 1, with different magnitudes and directions of field strength at points on the surface.

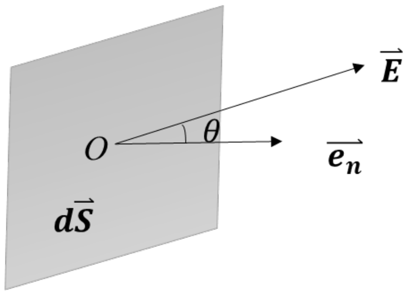

Figure 1: Illustration of the concept of electrostatic field fluxes

Dividing the surface into an infinite number of surface elements $d\vec{S}$ (differential) is finished, and the strength of the electric field at the surface elements can be considered to be constant, then the electric flux through each surface element can be expressed as:

$$d\Psi_e = \vec{E} \cdot d\vec{S} \tag{2}$$

The electric flux Ψ_e of the entire surface is then equal to the algebraic sum of the electric fluxes on all surface elements (the area division, which embodies the idea of integration), which can be denoted as:

$$\Psi_e = \iint_S d\Psi_e = \iint_S \vec{E} \cdot d\vec{S} \tag{3}$$

In the above expression, $\iint$ is the area fraction sign. If the surface is closed and the space are divided into inner and outer parts like a soccer ball, the electric flux can be expressed as:

$$\Psi_e = \oiint_S d\Psi_e = \oiint_S \vec{E} \cdot d\vec{S} \tag{4}$$

In the mentioned expression, $\oiint_S$ denotes integration along the entire closed surface.

Gauss's theorem in electrostatic fields is based on the premise of the validity of Coulomb's law, which reads, The electric field strength $\vec{E}$ flux Ψ_e of any closed surface S in an electrostatic field is equal to the algebraic sumΣq_{in} divided by ε_0 of the charges surrounded by the surface, independently of the charges outside the closed surface. The mathematical form is expressed as:

$$\Psi_e = \oiint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \Sigma q_{in} \tag{5}$$

If the charge is continuously distributed on the charged body, then Gauss's theorem can be expressed as

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$$\Psi_e = \oiint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho_e dV \tag{6}$$

Eq. (5) and (6) are both known as integral expressions of Gauss's theorem for electrostatic fields. The difference lies in that Eq. (5) is used for the case of discrete charge distributions, therefore, the charge enclosed by the surface uses the cumulative summation symbol Σ when solving for the electric flux; whereas, Eq. (6) is applied to the case of a charged body with a continuous distribution of charge. The volume division symbol $\iiint_V$ is used in Gauss's theorem for calculating the charge enclosed by the surface . The integral expression of Gauss's theorem for an electrostatic field reflects, on a macroscopic scale, the overall relationships between the field and the source of an electrostatic field, i.e., that is, the relationships between the total electric flux on an arbitrary closed surface (expressed here as a closed-surface integral of the electric field strength) and the total charge within the surface.

Gaussian theorem for electrostatic fields is formally similar to mathematical Gaussian theorem, but should not be confused. It can be considered that the electrostatic field provides a practical model for the mathematical Gaussian theorem, thus, the Gaussian theorem for electrostatic fields is a generalized application of the Gaussian theorem for vector analysis in higher mathematics.

3.2. Application of Differential to Gauss's Theorem in Electrostatic Fields with Scattering

Eq. (6) combined with Eq. (1) of Gauss theorem for vector analysis in higher mathematics, we have:

$$\oiint_S \vec{E} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{E} dV \tag{7}$$

Eq. (7) realizes the transformation of the closed surface integral of the electric field strength for an electrostatic field to a volume partition of the scatter of the electric field strength in the volume enclosed by the surface. The association (6) and (7) gives:

$$\iiint_V \nabla \cdot \vec{E} dV = \frac{1}{\varepsilon_0} \iiint_V \rho_e dV \tag{8}$$

Equation (8) can be applied universally in an electrostatic field, and gives the relationship between the volume fraction of the scattering of the electric field strength in an electrostatic field and the amount of charge carried by a charged body by means of the volume fraction explicitly. Then, the volume fraction is obtained by simultaneously withdrawing the volume fractions on both sides of the equation of (8):

$$\nabla \cdot \vec{E} = \frac{\rho_e}{\varepsilon_0} \tag{9}$$

Eq. (9) is the differential expression for Gauss's theorem for an electrostatic field, which shows more clearly that the dispersion of the electric field strength vector is equal to the charge density divided by ε_0.

However, for beginners, understanding the differential expression (6) for Gauss's theorem for an electrostatic field is a challenge for the following two reasons. The first one is understanding of the Laplace operator ∇, a differential operator, whose introduction reflects the application of differential ideas to Gauss's theorem. The operator ∇, often used to represent the gradient and scattering, represents full differentiation in all directions of space, which is equivalent to taking partial derivatives of a function in all orthogonal directions and multiplying them by the unit direction vector in each direction. For example, in three dimensions, the vector $\vec{A}(x, y, z) = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$, thus the scattering of vector $\vec{A}(x, y, z)$can be expressed in the form of equation (10).

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{10}$$

The second is the comprehension of the concept of scattering. Now that the concept of scattering directly affects the learning and understanding of Gaussian theorem for electrostatic fields, it is seriously necessary to further interprete scattering. The scattering of a vector $\vec{A}$, expressed by $\nabla \cdot \vec{A}$ as the pointwise multiplication of Laplace operator by the vector $\vec{A}$, is defined as the bulk density of the flux of the vector $\vec{A}$. And the flux of the vector $\vec{A}$ is defined as the area fraction of the vector $\vec{A}$. Thus the scatter can be written in the form of Eq. (11):

$$\nabla \cdot \vec{A} = \lim_{\Delta S \to 0} \frac{\oiint_S \vec{A} \cdot d\vec{S}}{\Delta V} \tag{11}$$

Therefore, associating Eqs. (4), (5), and (8), the dispersion of the electric field strength of the

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electrostatic field can be expressed as

$$\nabla \cdot \vec{E} = \lim_{\Delta S \to 0} \frac{\oint\!\!\oint_S \vec{E} \cdot d\vec{S}}{\Delta V} = \lim_{\Delta V \to 0} \frac{\iiint_V \nabla \cdot \vec{E} dV}{\Delta V} = \lim_{\Delta V \to 0} \frac{\frac{1}{\varepsilon_0} \iiint_V \rho_e dV}{\Delta V} = \frac{1}{\varepsilon_0} \lim_{\Delta V \to 0} \frac{\iiint_V \rho_e dV}{\Delta V} = \frac{1}{\varepsilon_0} \frac{d(\iiint_V \rho_e dV)}{\mathrm{d}V} = \frac{\rho_e}{\varepsilon_0} \quad (12)$$

Eq. (12), which reflects the intrinsic connection between the differential expression and the integral expression of Gauss's theorem for electrostatic fields, is a further interpretation of Eq. (9). Reflecting the pass of the electric field strength of the electrostatic field illustrates the relationship between the field and the field source of the electrostatic field, stating that the electrostatic field is an active field. And the field source could be written as: the electric charge volume density at rest with respect to the observer is equal to the ratio of electric charge volume density to ε_0. Also, the relationship between the field and the field source in an electrostatic field is illustrated from a microscopic point of view, showing that the electrostatic field is an active field and the field source is the electric charge that is stationary with respect to the observer.

3.3. The role of calculus for Gauss' theorem in electrostatic fields

The integral form of Gauss's theorem for electrostatic fields describes a physical quantity of macroscopic properties of the electrostatic field, the electric flux, which reflects macroscopically the overall relationship between the field and the source of the field in the electrostatic field, the total electric flux on an arbitrary closed surface (represented here as a closed surface integral over the electric field strength) and the total charge in the surface. And the differential form of Gauss's theorem of the electrostatic field describes a physical quantity of the microscopic nature of the electrostatic field, scattering, reflecting the relationship between the bulk density of the electric field intensity flux and the electric charge, and illustrating the relationship between the field and the source of the field in the electrostatic field from a microscopic point of view, the relationship between the divergence of electric field and charged density at any given point in the electrostatic field.

To sum up, the integral form of the Gauss theorem of the electrostatic field integrates the differential form, which will be simplified by Gauss formula later to describe the relationship between a certain spatial range of charge (charge density obtained in the space of the integral) and its surface electric flux (divergence of electric field in the space of the integral obtained by Gaussian formula to simplify the relationship). Both the differential and integral forms in Gauss's theorem for an electrostatic field reflect the relationship between the field and the field source in an electrostatic field, that is, the electrostatic field is an active field, and the source of the field is the charge that is stationary with respect to the observer.

4. Calculus in Gauss's Theorem for Teaching and Learning

4.1. Emphasize the transformation of physical models to mathematical models in teaching and learning

Using mathematical language to describe the laws and theorems of physics is a necessary ability for physics majors. The practical physical problems could be solved through establishing physical models, being abstracted, then being transformed into mathematical models, and using simple and clear mathematical expressions to express its meanings finally, which is not only conducive to the solution of the problem, but also improves learners' ability to apply advanced mathematics in the teaching process, especially the application of calculus in the course of Electromagnetism and other courses. Meanwhile, it contributes to the formation of a physical thinking mode of calculus in higher mathematics during the teaching process. In this way, students' understanding of course content such as Electromagnetism can be deepened, and the application of calculus in solving practical problems can be enhanced.

4.2. Integrate the idea of "constant variation" of calculus into physics teaching.

According to the ideas and methods of differentiation, any change of time and space in physics can be transformed into a collection of infinite microelements, and each microelement can be regarded as constant, which embodies the "constant variable" transformation idea of differentiation. This is different from the elementary mathematical thinking acquired before, which makes it more difficult for students to accept and master the application of calculus in courses such as Electromagnetics. Therefore, it is necessary to deeply integrate the "constant variation" idea of calculus into the process of physics teaching, accompanied with sufficient training in higher mathematics thinking and a long period of subtle influence, which can gradually change students' elementary mathematical thinking habits in the end. These can help to achieve the integration of calculus ideas and methods with the knowledge and problem solving of Electromagnetism and other courses.

In the teaching process, equal importance should be attached to the teaching of calculus and Electromagnetism and other courses, and the expansion of synergistic education should be strengthened to enhance the learners' learning effect of calculus and physics, which can not only solve relevant problems but also deepen students' understanding in the connotation of calculus. In this way, while completing the teaching, students also complete the learning and practicing of calculus theory, which in turn expands the learning outcomes.

4.3. Transform the concept of education and instruction from teaching task to talent cultivation

The change of subjective consciousness of teachers in teaching activities will have a great impact on the teaching effect. A teacher, taking the overall development of individual students as a starting point, is supposed to get rid of pure physical knowledge teaching, improve students' ability to understand knowledge and think independently, and strengthen the training of logical thinking. Realizing the conceptual transformation of education and instruction from teaching tasks to talent cultivation enables students to obtain new knowledge, new ideas and new methods. Meanwhile, their inherent thinking stereotypes and cognitive conflicts are sublimated and resolved.

5. Conclusion

The ideas and methods of calculus are fundamental ones in learning and solving physical problems (e.g., Electromagnetism). Based on the exploring the application of calculus to Gauss's theorem in electrostatic fields, the integral and differential expressions of Gauss's theorem for electrostatic fields are described in detail. The integral form and differential form of Gauss's theorem expression for electrostatic field show the relationship between the field and the field source of electrostatic field from a macroscopic point of view and a microscopic point of view respectively,that is, the electrostatic field is an active field. Meanwhile, the mutual transformation of the integral and differential expressions of Gauss's theorem in the electrostatic field further illustrates the intrinsic connection between integration and differentiation in advanced mathematics, that is, differentiation is the inverse operation of integration.

In addition, the discussion of the application of calculus to Gauss's theorem in electrostatic fields inspires the teaching of physics courses (e.g., electromagnetism) as well. Firstly, the transformation of physical models to mathematical models should be emphasized in teaching to promote learners' higher mathematical thinking ability and logical thinking ability. Moreover, integrating the "constant variation" idea of calculus into the teaching of physics and expanding the content of cooperative education are conductive to improving learners' learning effect on calculus and physics knowledge. Finally, teachers gradually practise changing teaching concepts from teaching tasks to talent cultivation, which will lead to acquiring new knowledge, new ideas and new methods and sublimating learners' inherent thinking and cognition.

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