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Further Exploration of Pythagorean Theorem

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ABSTRACT:*Pythagorean theorem was raised by Pythagoras in 551 BC, and it was also proposed in an even more ancient time by a Chinese mathematician called Shang Gao in the 11th century BC, although this process is not well-known. Moreover, it is one of the most famous theories, almost all of us have studied in our middle schools. The Pythagorean theorem tells us the relationship between three sides of a right triangle, that is, a2+b2=c2. However, mathematics is developing all the time, obviously, this theorem is not enough for us to explore in a higher dimension, and we are also interested in some equations with specific numbers. In this essay, we will focus on four questions, to generalize the Pythagoras theorem into three dimensions and discuss both rational and irrational solutions.*

KEYWORDS: *Pythagorean Theorem, stereographic projection, modular arithmetic*

1.Introduction

To this day, the Pythagorean theorem remains the most important single theorem in the whole of mathematics, said Jacob Bronowski in The Ascent of Man [1]. The theorem itself is known by various names, the hypotenuse theorem, the theorem of Pythagoras， and so on. There are now about 500 ways to prove the Pythagorean theorem, it is one of the theorems with the most proving methods. Pythagoras, an ancient Greek mathematician proved the theorem in the sixth century BC, but he was not the first person to discover the equation a2 + b2 =c2. Around 1600 BC, The Ancient Babylonians were aware of the Pythagorean number, and a clay tablet named Plimpton No.322 collected in Columbia University which was donated by G. Plimpton[2], showed a set of Pythagorean triple integers, although some of them were wrong (ancient people may make mistakes as well). Besides, the ancient Egyptians also used the Pythagorean

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theorem in building their great pyramids around 4500 years ago. Pythagorean triples were also mentioned in ancient Chinese books. Zhou Bi Suan Jing, China's oldest astronomy and mathematics work, about to be written in the 1st century BC, recorded a mathematician in Western Zhou Dynasty, Shang Gao, also mentioned Pythagorean Triples.

We say that the integers A, B, and C form a Pythagorean triple, if $A^2+B^2=C^2$. However, any multiple of A, B, and C can be another Pythagorean triple. So the idea of primitive triple arises, which says that A, B, C form a primitive triple if $A^2+B^2=C^2$ but there is no integer d which divides all A, B, and C other than ± 1. Similarly, we have a definition of primitive quadruples for equation $A^2+B^2+C^2=D^2$. We will discuss it in the first section below. Subsequently, we will illustrate the rational solution of $Ax^2+By^2=1$ under certain conditions. In the end, whether the equation $x^2+y^2=3$ have a rational solution will be shown.

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2. Pythagorean theorem in three dimensions

In this part, we want to find primitive quadruples (a, b, c, d), which satisfy the equation a2+b2+c2=d2.

Firstly, we can convert this equation into another form, that is, $x^2+y^2+z^2=1$, where all of x, y, and z are rational number, and then, we can connect the point A(0,0,1) and a random coordinate B(m, n, 0), where m and n are also rational, on the x-y plane to obtain a line l1 intersecting the sphere at a new point C. This process is shown in the graph below. (Figure 1)

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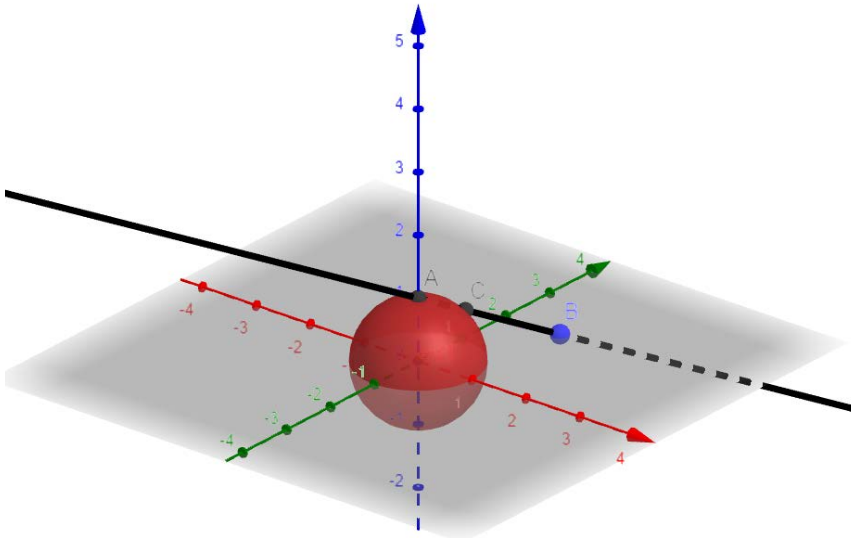

Figure 1

With the coordinates of A and B, we can calculate the equation of l1:

Let $t = \frac{x}{m} = \frac{y}{n} = \frac{z-1}{-1}$, then we can get that

$$\begin{cases} x = mt \\ y = nt \\ z = 1 - t \end{cases}$$

Put this into the equation $x^2 + y^2 + z^2 = 1$, we can see that

$$(mt)^2 + (nt)^2 + (1-t)^2 = 1$$

And solve this, we can obtain that

$$t = \frac{2}{m^2 + n^2 + 1}$$

To put the value of t back to x, y, and z, the results are:

$$x = \frac{2m}{m^2 + n^2 + 1}$$

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$$y = \frac{2n}{m^2 + n^2 + 1}$$

$$z = \frac{m^2 + n^2 - 1}{m^2 + n^2 + 1}$$

Because both m and n are rational numbers, so we can assume $m = \frac{\alpha}{\beta}$, and $n = \frac{\gamma}{\delta}$, where α and β are relatively prime positive integers, so are γ and δ.

Substituting m and n with $\frac{\alpha}{\beta}$ and $\frac{\gamma}{\delta}$ respectively, then

$$x = \frac{2\alpha\beta\delta^2}{\alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2}$$

$$y = \frac{2\gamma\delta\beta^2}{\alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2}$$

$$z = \frac{\alpha^2\delta^2 + \beta^2\gamma^2 - \beta^2\delta^2}{\alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2}$$

Putting these results back to the equation $x^2 + y^2 + z^2 = 1$ gives

$$\frac{(2\alpha\beta\delta^2)^2}{(\alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2)^2} + \frac{(2\gamma\delta\beta^2)^2}{(\alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2)^2} + \frac{(\alpha^2\delta^2 + \beta^2\gamma^2 - \beta^2\delta^2)^2}{(\alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2)^2} = 1$$

Finally, we can get our result by multiplying $(\alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2)^2$ at both sides, so the primitive quadruples are:

$$a = 2\alpha\beta\delta^2$$

$$b = 2\gamma\delta\beta^2$$

$$c = \alpha^2\delta^2 + \beta^2\gamma^2 - \beta^2\delta^2$$

$$d = \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2$$

where α and β are relatively prime numbers, γ and δ are relatively prime numbers.

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3.Rational solutions of $Ax^2 + By^2 = 1$ under certain conditions

In this section, we will try to show that there will be infinitely many rational solutions of the equation $Ax^2 + By^2 = 1$, if it has one rational solution, where A and B are non-zero rational numbers.

Similarly, we can draw a graph for this equation. (Figure 2)

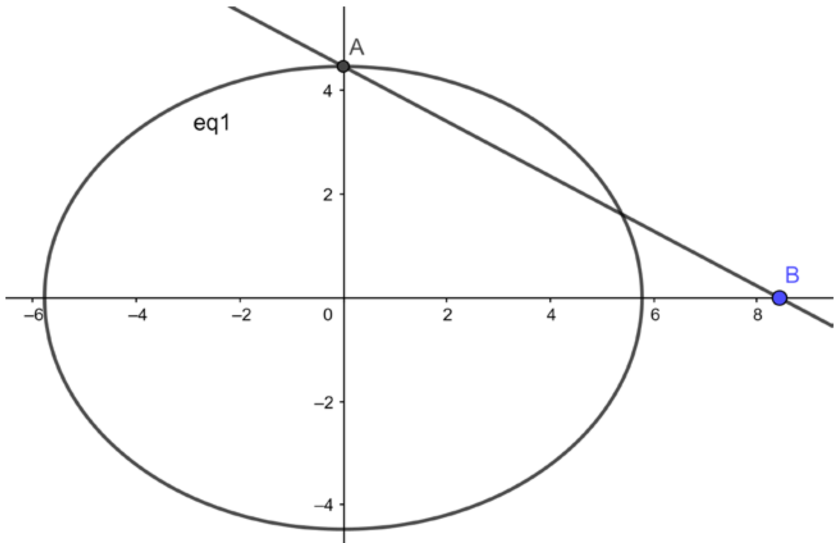

Figure 2

From the original equation $Ax^2 + By^2 = 1$, we can know that the ellipse will intersect the positive y-axis at a point A(a,b), and then, we can connect this point A to another rational point B, which is on the x-axis.

Let's assume the coordinate of B is (m,0), and we will know that the equation of AB is

$$y = -\frac{b}{m-a}x + \frac{bm}{m-a}$$

By substituting the y into the equation $Ax^2 + By^2 = 1$, we can obtain that

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$$Ax^2 + B\left(-\frac{b}{m-a}x + \frac{bm}{m-a}\right)^2 = 1$$

After solving this equation, we can get the other point C where line AB intersects the ellipse is

$$(\frac{2Bb^2m}{A(a-m)^2 + Bb^2} - a, \frac{2Bb^3m}{(a-m)[A(a-m)^2 + Bb^2]} - \frac{b(a+m)}{a-m})$$

Where A, B, a, b, m is all rational.

From the form of point C, we can conclude that C will be rational as long as m is rational. Because m can be any value as a result of B is an arbitrary point on the positive x-axis, there is infinitely many rational numbers form. Therefore, there will be infinitely many rational solutions of the equation $Ax^2 + By^2 = 1$.

4. Rational solutions of $x^2 + y^2 = 3$

Subsequently, we are going to prove that the equation $x^2 + y^2 = 3$ does not have a rational solution, and this is also related to the next section. In this section, we will prove by contradiction.

At first, we can assume that this equation has rational solutions, then we can make $x = \frac{b}{a}$, and $y = \frac{d}{c}$, where it is relatively prime for both a, b, c and d. Then, we can change the equation into

$$\frac{b^2}{a^2} + \frac{d^2}{c^2} = 3$$

By multiplying a^2c^2 at each side, we can get that

$$b^2c^2 + a^2d^2 = 3a^2c^2$$

Compare b^2c^2 and $3a^2c^2$, because a and b are relatively prime, a and b do not have common factors. As a result, b has to have factor 3, to make sure the equation can be correct. After this, we can assume make $b = 3m$.

Similarly, by taking the same process to a^2d^2 and $3a^2c^2$, we can also know that d is a multiple of 3, and make $d = 3n$.

By substituting $b = 3m$ and $d = 3n$ to the equation $b^2c^2 + a^2d^2 = 3a^2c^2$, we can have a new equation

$$9m^2c^2 + 9n^2a^2 = 3a^2c^2$$

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By dividing 3 at each side, we will get

$$3(m^2c^2 + n^2a^2) = a^2c^2$$

From this equation, we can see that either a or c has to be divisible by 3, which means one of a and c must have the common factor 3 with their corresponding denominator, that is, b or d.

We can find that this result is contradicted to our assumption, which is that both a, b, c and d, are relatively prime, so x and y cannot be written in the form of fractional numbers, in other words, both of them are not able to be rational in the equation $x^2 + y^2 = 3$, then, this equation has no rational solution.

5. Rational solutions of $x^2 + y^2 = d$

In the final part, we will continue the topic in part 4, and do some further explorations on it. The question is, investigate for which integers d can $x^2 + y^2 = d$ have a rational solution. It is easy to observe that if d is a square, then d can be written as the sum of two squares.

Since we can make $d = n^2$, where n is an integer. By transforming this equation, we will see

$$\left(\frac{x}{n}\right) + \left(\frac{y}{n}\right)^2 = 1$$

Using the fact of primitive triples, we know that both $\frac{x}{n}$ and $\frac{y}{n}$ can be infinitely many rational numbers. Therefore, the equation $x^2 + y^2 = d$ have a rational solution.

In conclusion, if d is a squared number, then the equation $x^2 + y^2 = d$ have a rational solution. But it is not the whole answer.

Conclusions

This project provides suitable ways, including stereographic projection and modular arithmetic, for several problems derived from the Pythagorean Theorem.

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