ISSN 2706-655X Vol.4, Issue 5: 64-68, DOI: 10.25236/IJFET.2022.040511

Physical Application of Taylor Formula

Daohong Liu[1], Chenyuan Tian[2]

[1]*College of Mathematics and Physics, Chengdu University of Technology, Chengdu, Sichuan, China*
[2]*College of Mathematics and Physics, China University of Geosciences (Wuhan), Wuhan, Hubei, China*

Abstract: *Taylor's formula is an important content in advanced mathematics, and it is also an indispensable mathematical tool in the study of function limits and error estimation. In this paper, Taylor's formula is used to deduce the electric potential energy of the charge system in the applied electric field and to prove the Noether's theorem of the point particle system under the action of conservative force. Finally, the expression of potential energy of multi electrode in external electric field and the reason why Noether's theorem is established in conservative system are given, and the superposition principle of potential energy of point particle system is indirectly explained.*

Keywords: *Taylor formula; Charge system; Noether's theorem; Conservative system*

1. Introduction

Taylor's formula is a formula that uses the information of a function at a point to describe the value near it. As an important theoretical tool for analyzing and studying the limit of functions, Taylor's formula can simplify complex problems, meet high accuracy and precision, and can be applied to many mathematical fields, and provides strong support for the rapid development of modern calculus. Therefore, Taylor's formula also has important application value in the field of physics.

At present, the electrodynamics textbooks used by most domestic colleges and universities do not give the detailed derivation process of the expression of electric potential energy of multi electrode in the applied electric field, and the proof process of Noether's theorem, which is particularly important in the conservative system, is cumbersome in the study of theoretical mechanics [1-3]. The simple proof process makes many students have a headache when learning physics. Therefore, the purpose of this paper is to give the detailed derivation process of the expression of electric potential energy of multi electrode in the applied electric field, and use the idea of Taylor formula to prove the reason why Noether's theory is established in the conservative system.

In this paper, the derivation process is divided into two parts. The first part is the elaboration of relevant concepts and the preparation of theoretical derivation. The second part is the derivation of Taylor formula.

2. Taylor formula

2.1. Taylor formula of univariate function

Let the function $f(x)$ have a continuous derivative of $(n+1)$ order in the open interval (a,b) containing x_0, then when $x \epsilon (a,b)$:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x) \quad (1)$$

Where, $R_n(x)$ is the Langrange Remainder [4]. Then $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, where ξ is between x and x_0.

2.2. Taylor formula of multivariable function

Let the function $f(x,y,z)$ be continuous in a neighborhood of point (x_0, y_0, z_0) and have a continuous partial derivative of order $(n+1)$, (x_0+u, y_0+v, z_0+w) at any point in this neighborhood, then

ISSN 2706-655X Vol.4, Issue 5: 64-68, DOI: 10.25236/IJFET.2022.040511

$$f(x_0+u,y_0+v,z_0+w)=f(x_0,y_0,z_0)+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)f(x_0,y_0,z_0)$$
$$+\frac{1}{2!}\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)^2 f(x_0,y_0,z_0)+\cdots \tag{2}$$

For the Taylor expansion of multivariable function, its form is similar to that of univariate function, but it will be more complex in operation and representation [5].

3. Application of Taylor formula

3.1. Electric potential energy of charge system in external electric field

3.1.1. Preparation

If we want to study the distribution of charge in the research object, we must deduce the information about charge distribution by calculating its interaction with the applied electric field. We assume that we know the potential distribution function of the external electric field, and our strategy is to express the charge distribution information of the position as the superposition of the basic charge distribution. Each basic charge distribution can only interact with the k order in the Taylor expansion of the potential.

3.1.2. Derivation process

Let $u(x,y,z)$ represent the potential distribution function of the known external electric field, and ρ_e represent the charge density function in the sample at the origin and the sample volume is tiny. The electric potential energy of the sample under the action of the applied electric field potential W can be expressed by the following formula,

$$W=\int dV\big(\rho_e u(x,y,z)\big) \tag{3}$$

Then, we use the Taylor formula to express the function $u(x,y,z)$ as the Taylor expansion at the coordinate origin,

$$u(x,y,z)=u(x_0,y_0,z_0)+\left[x\left(\frac{\partial u}{\partial x}\right)_0+y\left(\frac{\partial u}{\partial y}\right)_0+z\left(\frac{\partial u}{\partial z}\right)_0\right]$$
$$+\frac{1}{2}\left[x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_0+y^2\left(\frac{\partial^2 u}{\partial y^2}\right)_0+z^2\left(\frac{\partial^2 u}{\partial z^2}\right)_0+2xy\left(\frac{\partial^2 u}{\partial x\partial y}\right)+2xz\left(\frac{\partial^2 u}{\partial x\partial z}\right)+2yz\left(\frac{\partial^2 u}{\partial y\partial z}\right)\right] \tag{4}$$

The first term of total energy is:

$$W_1=\int dV\big(\rho_e u(x_0,y_0,z_0)\big)=Qu_0 \tag{5}$$

In the expansion, this term indicates that the basic distribution of charge in the sample is the point charge at the coordinate origin, that is, the monopole.

For the second item requiring total energy, we first simplify it as follows:

$$x\left(\frac{\partial u}{\partial x}\right)_0+y\left(\frac{\partial u}{\partial y}\right)_0+z\left(\frac{\partial u}{\partial z}\right)_0=\boldsymbol{r}\cdot(\boldsymbol{\nabla}u)_0=-\boldsymbol{r}\cdot\boldsymbol{E_0} \tag{6}$$

Where $\boldsymbol{E_0}$ represents the electric field intensity generated by the potential u at the coordinate origin, and $\boldsymbol{r}$ represents the direction vector. Then, the potential energy integral of this part is:

$$W_2=-\boldsymbol{E_0}\int dV(\rho_e\boldsymbol{r})\equiv-\boldsymbol{E_0}\cdot\boldsymbol{p} \tag{7}$$

Where, the integral of $\rho_e\boldsymbol{r}$ to the sample volume is called the electric dipole moment in the sample, and recorded as $\boldsymbol{p}$. This means that there is a separate positive and negative charge in the sample. Along the electric dipole formed by passing through the coordinate origin, the direction is from negative charge to positive charge [6].

For the solution of the third term, which is also the focus of this part, we start with the separation of mixed partial derivatives.

$$2xy\left(\frac{\partial^2 u}{\partial x\partial y}\right)=xy\left(\frac{\partial^2 u}{\partial x\partial y}\right)+yx\left(\frac{\partial^2 u}{\partial y\partial x}\right) \quad (8)$$

Therefore, the third term of the Taylor expansion of the charge distribution function can be expressed as:

$$\frac{1}{2!}\left[\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^2 u\right]_0$$

$$=\frac{1}{2}[x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_0+y^2\left(\frac{\partial^2 u}{\partial y^2}\right)_0+z^2\left(\frac{\partial^2 u}{\partial z^2}\right)_0+xy\left(\frac{\partial^2 u}{\partial x\partial y}\right)_0+yx\left(\frac{\partial^2 u}{\partial y\partial x}\right)_0$$

$$+xz\left(\frac{\partial^2 u}{\partial x\partial z}\right)_0+zx\left(\frac{\partial^2 u}{\partial z\partial x}\right)_0+yz\left(\frac{\partial^2 u}{\partial y\partial z}\right)_0+zy\left(\frac{\partial^2 u}{\partial z\partial y}\right)_0 \quad (9)$$

Since there is no charge generating the potential function u in sample V, u obeys the Laplace equation at the coordinate origin:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_0+\left(\frac{\partial^2 u}{\partial y^2}\right)_0+\left(\frac{\partial^2 u}{\partial z^2}\right)_0=(\boldsymbol{\nabla}^2 u)_0=0 \quad (10)$$

This means that we can add $(\boldsymbol{\nabla}^2 u)_0$ to the second term of Taylor expansion without changing the value of the expression. Thus, the form of the original term can be rewritten as:

$$\frac{1}{6}[(3x^2-r^2)\left(\frac{\partial^2 u}{\partial x^2}\right)_0+3xy\left(\frac{\partial^2 u}{\partial x\partial y}\right)_0+3xz\left(\frac{\partial^2 u}{\partial x\partial z}\right)_0$$

$$+3yx\left(\frac{\partial^2 u}{\partial y\partial x}\right)_0+(3y^2-r^2)\left(\frac{\partial^2 u}{\partial y^2}\right)_0+3yz\left(\frac{\partial^2 u}{\partial y\partial z}\right)_0$$

$$+3zx\left(\frac{\partial^2 u}{\partial z\partial x}\right)_0+3zy\left(\frac{\partial^2 u}{\partial z\partial y}\right)_0+(3z^2-r^2)\left(\frac{\partial^2 u}{\partial z^2}\right)_0] \quad (11)$$

Where $r^2=x^2+y^2+z^2$. Then, we define function α_{ij}, whose expression iswhose expression is:

$$\alpha_{ij}=3ij-\delta_{ij}r^2 \quad (12)$$

Where δ_{ij} is the Kronecker delta function. Write the second-order term as a summation:

$$\frac{1}{6}\sum_{i=x,y,z}\sum_{j=x,y,z}\alpha_{ij}\left(\frac{\partial^2 u}{\partial i\partial j}\right)_0 \quad (13)$$

Hence, we get the electric potential energy of the point quadrupole at the origin of the coordinate in the external electric field. The total potential energy of the sample in the external electric field is written in the following form:

$$W=W_1+W_2+W_3+\cdots \quad (14)$$

3.2. Noether's theorem and Translational symmetry

3.2.1. Basic concepts

Noether's theory is an important result in theoretical physics. It describes that the behavior of a point particle system does not change under a specific infinitesimal transformation, then this transformation corresponds to a conserved quantity [7].

3.2.2. Derivation route

Then, we try to use Taylor formula to simply deduce Noether's theory. For simplicity, we assume that the system is conservative, that is, the system is not affected by external forces.

Firstly, we assume that the system consists of N point particle systems, the position of each particle is recorded as (x_i, y_i, z_i), and the potential energy function of the system after a minimal displacement in the xOy plane is

$$U([x_1+\delta, x_2+\delta,\cdots,x_n+\delta],[y_1+\varepsilon,y_2+\varepsilon,\cdots,y_n+\varepsilon],Z) \quad (15)$$

ISSN 2706-655X Vol.4, Issue 5: 64-68, DOI: 10.25236/IJFET.2022.040511

Where δ and ε are the minimum distance of translation in the x direction and the minimum distance of translation in the y direction, respectively.

We carry out linear Taylor expansion on the above functions, and we get:

$$U([x_1+\delta, x_2+\delta, \cdots, x_n+\delta], [y_1+\varepsilon, y_2+\varepsilon, \cdots, y_n+\varepsilon], Z) \\ = U([x_1, x_2, \cdots, x_n], [y_1, y_2, \cdots, y_n], Z) \\ +\delta\left(\frac{\partial U}{\partial x_1}+\frac{\partial U}{\partial x_2}+\cdots+\frac{\partial U}{\partial x_n}\right)+\varepsilon\left(\frac{\partial U}{\partial y_1}+\frac{\partial U}{\partial y_2}+\cdots+\frac{\partial U}{\partial y_n}\right) \tag{16}$$

According to the translational symmetry of mechanical energy to the spatial coordinate system, we can know that [8]

$$\begin{cases} \frac{\partial U}{\partial x_1}+\frac{\partial U}{\partial x_2}+\cdots+\frac{\partial U}{\partial x_n} \equiv 0 \\ \frac{\partial U}{\partial y_1}+\frac{\partial U}{\partial y_2}+\cdots+\frac{\partial U}{\partial y_n} \equiv 0 \end{cases} \tag{17}$$

So far, we have obtained the simplest mathematical expression of Noethe's theorem:

$$U([x_1+\delta, x_2+\delta, \cdots, x_n+\delta], [y_1+\varepsilon, y_2+\varepsilon, \cdots, y_n+\varepsilon], Z) \\ = U([x_1, x_2, \cdots, x_n], [y_1, y_2, \cdots, y_n], Z) \tag{18}$$

For a conservative system, the force acting on it is calculated by the following formula,

$$F = -\boldsymbol{\nabla} U \tag{19}$$

Therefore, the vector sum of the $\boldsymbol{x}$ component and the vector sum of the $\boldsymbol{y}$ component of the force acting on each particle are zero. This means that there is no other external force on the system in the $\boldsymbol{x}$ and $\boldsymbol{y}$ directions.

Finally, according to the relationship between force and momentum, we can see that the components of momentum in $\boldsymbol{x}$ direction and $\boldsymbol{y}$ direction are conserved.

Note: The equivalence between using Taylor formula to prove Noether 's theorem and Lagrange function to prove the theorem remains to be studied. Here is only one idea.

4. Conclusion

In this paper, the expression of potential energy of charge system in external electric field and the simple proof of Noether's theorem in conservative system are given by using the method of Taylor formula.

It is necessary to note that when proving Noether's theorem, the potential energy function can also write higher-order Taylor expansion, but the second-order and above parts are very small in physics and can be ignored. Expanding to the first order is the embodiment of the approximate method in physics research, and can also be understood as the embodiment of the approximate calculation of Taylor formula in mathematics.

Although the linear approximation of physical quantities using Taylor formula is not a very important method in general physics, linear approximation or high-order approximation will become particularly important in learning theoretical physics. Therefore, it is very important to be familiar with and master Taylor formula, and I hope this paper can expand learning ideas for more students.

References

[1] Zhou Y B. Theoretical Mechanics [M]. Higher Education Press, 2009.
[2] Zhang X Z, Zhang H Q. Classical Electrodynamics [M]. Science Press, 2013.
[3] Yin Z. Electrodynamics [M]. Science Press, 2010.
[4] Department of Mathematics, Tongji University. Advanced Mathematics [M]. Higher Education Press, 2014.
[5] Wu C S. Mathematical physics method [M]. Higher Education Press, 2015.
[6] Guo S H. Electrodynamics [M]. Higher Education Press, 2008.

ISSN 2706-655X Vol.4, Issue 5: 64-68, DOI: 10.25236/IJFET.2022.040511

[7] Brading K, Brown H R. Noether's Theorems and Gauge Symmetries [J]. Arxiv Cornell University Library, 2000.
[8] Qi A S. Mechanics [M]. Higher Education Press, 2018.