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Proof of the Collatz Conjecture

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Abstract: *The Collatz Conjecture primarily explores whether a sequence of positive integers, generated by a specific iterative rule, ultimately converges to 1. The study of its proof is not only aimed at answering a specific mathematical question but also at delving into the essence and boundaries of mathematics, thereby advancing the development of mathematical science. The conjecture encompasses all positive integers, focusing on understanding and proving that when performing the following operations on any positive integer: if it is odd, multiply by 3 and add 1; if it is even, divide by 2, after a finite number of iterations, each sequence of operations inevitably reaches the number 1 and forms a trivial cycle: {4, 2, 1}. By introducing the concept of roots and utilizing constructive methods and mathematical induction, we explore and analyze related issues of the Collatz Conjecture, leading to an in-depth investigation that proves the conclusion that the Collatz Conjecture transforms any positive integer into 1.*

Keywords: *Collatz Conjecture; branch number; transformation symbols; transformation paths; roots*

1. Introduction

The Collatz Conjecture, commonly referred to as the 3*n* + 1 problem, states: For any positive integer $n, n \in \mathbb{Z}^+$, if *n* is even, repeatedly divide it by 2 until it becomes an odd number; if *n* is odd, multiply it by 3 and then add 1. This process is repeated indefinitely, and after a finite number of steps, one will inevitably reach 1. Its proposition is:

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

$\forall n \in \mathbb{Z}^+$, there exists a positive integer $k \in \mathbb{Z}^+$, and there is always $f^k(n) = 1$.

The Collatz Conjecture operates under very simple rules. Taking the number 13 as an example: since 13 is an odd number, multiplying it by 3 gives 39, and adding 1 results in 40. Dividing by 2 gives 20; since 20 is even, dividing by 2 again results in 10. Then, since 10 is even, dividing by 2 gives 5. Now, 5 is odd, so we multiply by 3, add 1, and halve the result to get 8. Since 8 is even, dividing by 2 gives 4; 4 is even as well, and dividing by 2 results in 2; finally, dividing 2 by 2 leads us to 1.

The Collatz Conjecture was proposed by German mathematician Lothar Collatz in the 1930s [1]. It states that regardless of which positive integer is chosen as the starting point, applying the aforementioned rules will ultimately lead to 1. However, the simplicity of the operations starkly contrasts with the difficulty of proving the conjecture. The famous mathematician Paul Erdős once remarked, "Mathematics is not ready for such problems" [2]. Jeff Lagarias also stated, "This is an exceptionally difficult problem that lies completely beyond the current scope of mathematics" [3]. To date, with the help of computer networks, all integers below 10201020 have been verified to converge to 1 [4]. The Collatz Conjecture is one of the most intriguing problems in mathematics and is also known by various names such as the odd/even conjecture, hailstone conjecture, 3*n* + 1 conjecture, and Kakutani conjecture, among others.

In 2019, the renowned mathematician Terence Tao published a paper proving that the conjecture holds for almost all positive integers [5]. For the sake of clarity in this article, non-negative integers will be represented as $\mathbb{N}$, and positive integers will be represented as $\mathbb{Z}^+$. Since any even positive integer will, through a series of Collatz transformations involving division by 2, eventually become an odd integer, we can focus solely on the sequences of transformations involving odd integers.

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2. Related Definitions and Conclusions

2.1 Odd Numbers

Integers that cannot be divided by 2 are called odd numbers. Positive odd integers can be represented as $2k+1,\ k \in \mathbb{N}$.

Positive odd integers of the form $4k+1$ can be further subdivided into three categories: $12k+1, 12k+5, 12k+9$. Similarly, $k \in \mathbb{N}$ can be subdivided into four categories: $16k+1, 16k+5, 16k+9, 16k+13$.

Positive odd integers of type $4k+3$ can be subdivided into: $12k+3$, $12k+7$, $12k+11$, $k \in \mathbb{N}$.

2.2 Branching Numbers

Positive odd integers that are divisible by 3 are referred to as branching numbers. These branching numbers can be represented as $3*(2k+1), k \in \mathbb{N}$. Characteristics of branching numbers include: branching numbers are terminal numbers, and they cannot undergo the inverse Collatz transformation.

2.3 Multiplicity of Integers

For an odd integer that belongs to type $12m+1$, $12m+5$, $12m+9$, it can also be classified as a type $16k+1$, $16k+5$, $16k+9$, $16k+13$ odd integer.

The relationship between positive odd integers of type $12m+1$, $12m+5$, $12m+9$ and positive odd integers of type $16k+1$, $16k+5$, $16k+9$, $16k+13$ is as follows:

$$16k+1\begin{cases}12m+1\\12m+5\\12m+9\end{cases},\qquad 16k+5\begin{cases}12m+1\\12m+5\\12m+9\end{cases}$$

$$16k+9\begin{cases}12m+1\\12m+5\\12m+9\end{cases},\qquad 16k+13\begin{cases}12m+1\\12m+5\\12m+9\end{cases}\quad k \in \mathbb{N}, m \in \mathbb{N}$$

2.4 Transformation Symbols

The Collatz transformation symbol is denoted as $\rightarrow$.

2.5 Transformation Path

The chain formed by connecting all the positive odd integers encountered during the Collatz transformation process using the Collatz transformation symbol is referred to as the Collatz transformation path.

The transformation path is denoted as: $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow ... \rightarrow x_n$

(1) Two Types of Transformation Paths.

① Existential Single Transformation Path:

$$\mathrm{x}_0 \rightarrow x_1 \rightarrow x_2 \rightarrow ... \rightarrow x_n$$

② Existential Composite Transformation Path:

$$\begin{array}{l} x_0 \to x_1 \to x_2 \to \ldots \to x_n \\ \quad\ \ \uparrow \\ \quad\ \ \lambda_0 \end{array}$$

③ Non-existent, Violating Uniqueness:

$$\begin{array}{l} x_0 \to x_1 \to x_2 \\ \qquad\quad \downarrow \\ \qquad\quad \lambda_0 \end{array}$$

(2) Two Important Properties of the Collatz Transformation Path: Uniqueness of the Transformation Path; Transitivity of the Transformation Path.

2.6 Image and Preimage

If the positive odd integer a is transformed into the positive odd integer b through one iteration of the Collatz operation, then b is called the image of a, and a is called the preimage of b.

(1) Descending Number: According to the definition of image and preimage, if $a > b$ is true, we call a a descending number.

(2) Ascending Number: According to the definition of image and preimage, if $a < b$ is true, we call a an ascending number.

2.7 Relativity of Image and Preimage

In the process of the Collatz transformation path, the concepts of image and preimage are relative.

For example: In the transformation path, $x_0 \to x_1 \to x_2 \to \ldots \to x_n$ and x_0 are the preimages of x_1, and x_1 is the image of x_0; x_1 is the preimage of x_2, and x_2 is the image of x_1; similarly, x_{n-1} is the preimage of x_n, and x_n is the image of x_{n-1}.

2.8 Mapping and Inverse Mapping

The Collatz mapping transformation is denoted by $\overrightarrow{C}$, and the inverse mapping transformation is denoted by $\overleftarrow{C}$.

2.9 Same Root

The same root is denoted by Y. If two positive integers A and B have at least one common number in their respective Collatz iteration processes, then these two numbers are said to have the same root, which is denoted as $A \mathrm{Y} B$. Having the same root implies that there is an intersection in their transformation sequences.

2.9.1 Rules of Same Root Operation

Self-same root rule: $A \mathrm{Y} A$ is the same root with any number as itself.

Same root equivalence rule: If $A \mathrm{Y} B$, then $B \mathrm{Y} A$ is in the same root relation. The same root relation is symmetric.

Same root transitive rule: If $A \mathrm{Y} B$ and $B \mathrm{Y} C$, then $A \mathrm{Y} C$ is in the same root relation. The same root relation is transitive.

2.9.2 Two Different Representations of the Same Root

Indirect same root: If A transforms into B through a single Collatz transformation ($A \to B$), then $A \mathrm{Y} B$. If C transforms into B through another single Collatz transformation ($C \to B$), then

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$C \mathrm{Y} B$. Thus, $A \mathrm{Y} C$ signifies that A and C are indirectly related through transformation rules and the concept of same root. Indirect same root can also be referred to as same-level same root, characterized by the ability to classify positive odd integers into layers based on the Collatz transformation rules, ensuring that odd integers within the same layer have the same transformation path. The composite transformation path is the combination of both indirect same root and direct same root.

Direct same root: If A transforms into B through a single Collatz transformation ($A \mathrm{Y} B$), then $A \mathrm{Y} B$. Here, A and B are directly related through transformation rules and the concept of same root. Direct same root can also be referred to as inter-level same root, characterized by the ability to associate same root sequences between different layers based on the reverse Collatz transformation rules. The single transformation path is the representation of direct same root.

2.9.3 Significance of Same Root

By introducing the concept of the same root, we can apply the Collatz transformation rules in reverse, allowing different types of positive odd integers, which were originally distinct, to be associated through the same root. This results in them sharing the same transformation path.

3. Lemmas and Corollaries

3.1 Lemma: $4k+1$ *Odd numbers are descending numbers,* $k \in \mathbb{Z}^+$

Proof: Since $\frac{(4k+1)\times 3+1}{4} = 3k+1 < 4k+1$, $k \in \mathbb{Z}^+$, When k is an odd number, $3k+1$ can also be divisible by 2, and the result after division will keep getting smaller, thus the conclusion holds.

Inference: Let m be an odd number of type $4k+1$ and $k \in \mathbb{N}$, and let a be transformed into an odd number b after one iteration of the Collatz operation through a continuous division by 2. According to Lemma 2.1, we have $m \geq 2$, $m \in \mathbb{Z}^+$. Conversely, let m be a positive integer odd number a that is transformed into an odd number b after one iteration of the Collatz operation through a continuous division by 2. If $m \geq 2$, then a is of types $4k+1$ and $k \in \mathbb{N}$, which can be proven by contradiction.

3.2 Lemma: Odd number $4k+3$ *is an ascending and descending number,* $k \in \mathbb{N}$

"Proof: Since $\frac{(4k+3)\times 3+1}{2} = 6k+5 > 4k+3$ and $k \in \mathbb{N}$, the conclusion holds."

"Corollary: Let m be a $4k+3$-type odd number and $k \in \mathbb{N}$ be an odd number a that becomes an odd number b after one Collatz iterative operation, with the number of times it is continuously divided by 2. From Lemma 2.2, we can obtain $m=1$ and $m \in \mathbb{Z}^+$. Conversely, let m be a positive integer odd number a that becomes an odd number b after one Collatz iterative operation, with the number of times it is divided by 2. If $m=1$, then a is $4k+3$, and $k \in \mathbb{N}$-type odd numbers can be proved by contradiction."

3.3 Lemma: For any $4k+3$, $k \in \mathbb{N}$ *-type odd number* x_0*, there exists a smallest positive integer* $n_0 \in \mathbb{Z}^+$ *such that the* n_0*-th Collatz mapping transformation must result in an* $4k+1$*-type odd number, denoted as* $f^{n_0}(x_0) = 4k+1, k \in \mathbb{N}$

Proof: Using the construction method, for any given one $4k+3$, $k \in \mathbb{N}$ "For an h-type odd number *x₀*, we can construct this odd number as follows." $k_0 \times 2^m \times 2^2 \times 3^n - 1$, $k_0 \in \mathbb{Z}^+$, $m \in \mathbb{N}$, $n \in \mathbb{N}$.

The construction steps are as follows: first, add 1 to get $4k+4$ and $k \in \mathbb{N}$, which can obviously be divided by 2^2. Then, continuously divide the resulting number by 3 to obtain 3^n. When it cannot be

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divided by 3 even once, let it be $n=0$. Next, continuously divide the resulting number by 2 to obtain 2^m. When it cannot be divided even once, let it be $m=0$. Finally, we find that the coefficients k_0 and k_0 are odd numbers. The proof is divided into two cases:

When $m=0$, $x_0 = k_0 \times 2^2 \times 3^n - 1$.

The first Collatz iterative operation: $x_1 = k_0 \times 2^1 \times 3^{n+1} - 1$ continuously divided by 2 only once;

The second Collatz iterative operation: $x_2 = k_0 \times 3^{n+2} - 1$ continuously divided by 2 only once;

Since k_0 is an odd number, it is evident that x_2 is an even number and can still be continuously divided by 2. Thus, x_1 can be transformed through the Collatz operation to obtain an odd number x_2, which can be continuously divided by 2 more than twice. From Corollary 2.1.1, it can be inferred that x_1 is an $4k+1$-type odd number. At this point, the smallest positive integer $n_0=1$ obtained confirms the proposition.

When $m>0$, $x_0 = k_0 \times 2^m \times 2^2 \times 3^n - 1$.

The first Collatz iterative operation: $x_1 = k_0 \times 2^{m-1} \times 2^2 \times 3^{n+1} - 1$ continuously divided by 2 only once.;

The second Collatz iterative operation: $x_2 = k_0 \times 2^{m-2} \times 2^2 \times 3^{n+2} - 1$ continuously divided by 2 only once.;

...

The *m*-th Collatz iterative operation: $x_m = k_0 \times 2^2 \times 3^{n+m} - 1$ continuously divided by 2 only once.;

The *m*+1-th Collatz iterative operation: $x_{m+1} = k_0 \times 2^1 \times 3^{n+m+1} - 1$ continuously divided by 2 only once.;

The *m*+2-th Collatz iterative operation: $x_{m+2} = k_0 \times 3^{n+m+2} - 1$ continuously divided by 2 only once.

Since k_0 is an odd number, it is clear that x_{m+2} is an even number, which can also be continuously divided by 2. Thus, x_{m+1} undergoes the Collatz transformation to yield the odd number x_{m+2}, which can be divided by 2 more than two times. By Corollary 2.1.1, it can be concluded that x_{m+1} is of classes $4k+1$ and $k \in \mathbb{N}$ as odd numbers. At this point, the minimum positive integer $n_0 = m+1$ is obtained, and the proposition is established.

In summary, the original proposition is established.

(1) Equivalent Proposition: For any odd positive integer of class $4k+1$, if the Collatz conjecture holds, then the Collatz conjecture must necessarily hold for any odd positive integer.

From Lemma 2.3, it can be inferred that any odd positive integers of classes $4k+3$ and $k \in \mathbb{N}$, after a finite number of Collatz mappings, must result in odd integers of classes $4k+1$ and $k \in \mathbb{N}$. Therefore, the equivalent proposition is established.

(2) Lemma: There are three types of Collatz transformation path models (see Figure 1-3) for odd positive integers of class $4k+3$.

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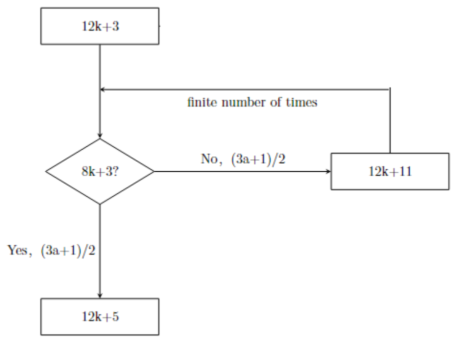

Figure 1: The First Transformation Path Model

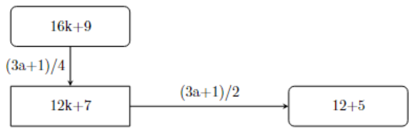

Figure 2: The Second Transformation Path Model

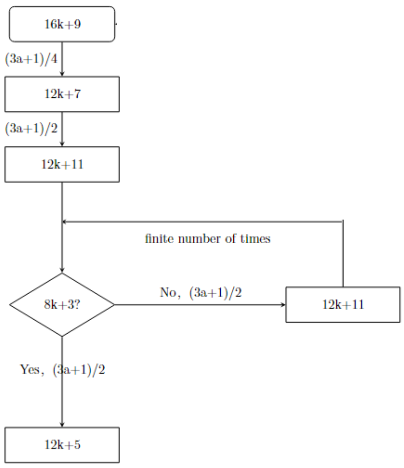

Figure 3: The Third Transformation Path Model

Prove that for positive integer odd numbers of classes $12k+3$ and $k \in \mathbb{N}$, using the Collatz mapping transformation formula $b = \frac{(3 \times a)+1}{2^r}$, let $a = 12k+3$ and $k \in \mathbb{N}$ be positive integer odd numbers. Then, there exists $b = 18k+5$. When k is an even number, $b = 36m+5$ is $12m+5$ and $m \in \mathbb{N}$ are positive integer odd numbers. When k is an odd number, $b = 36m+23$ is $12m+11$ and $m \in \mathbb{N}$ are positive integer odd numbers.

For positive integer odd numbers of classes $12k+7$ and $k \in \mathbb{N}$, using the Collatz mapping transformation formula $b = \frac{(3 \times a)+1}{2^r}$, let $a = 16k+9$ and $k \in \mathbb{N}$ be positive integer odd numbers. Then, there exists a positive integer odd number of class $b = 12k+7$. When $a = 12k+7$ and $k \in \mathbb{N}$ are positive integer odd numbers, using the Collatz mapping transformation formula $b = \frac{(3 \times a)+1}{2^r}$, there exists a positive integer odd number of class $b = 18k+11$. When k is an even number, $b = 36m+11$ is $12m+11$, and $m \in \mathbb{N}$ are positive integer odd numbers. When k is an odd number, $b = 36m+29$ is $12m+5$, and $m \in \mathbb{N}$ is a positive integer odd number.

For positive integer odd numbers of classes $12k+11$ and $k \in \mathbb{N}$, using the Collatz mapping transformation formula $b=\frac{(3\times a)+1}{2^r}$, let $a=12k+11$ and $k\in\mathbb{N}$ be positive integer odd numbers. Then, there exists a positive integer odd number of class $b=18k+17$. When k is an even number, $b=36m+17$ is $12m+5$, and $m\in\mathbb{N}$ is a positive integer odd number. When k is an odd number, $b=36m+35$ is $12m+11$, and $m\in\mathbb{N}$ is a positive integer odd number. According to Lemma 2.3, it is known that the positive integer odd numbers of classes $12m+11$ and $m\in\mathbb{N}$ will eventually lead to $12m+5$ and $m\in\mathbb{N}$, which are positive integer odd numbers, after a finite number of Collatz iterations. Based on this, the original proposition is established.

(3) Corollary: For any class $4k+3, k\in\mathbb{N}$ of positive integer odd numbers, there must exist a positive integer of class $12m+5, m\in\mathbb{N}$ that shares the same root.

Proof: Based on Lemma 2.3 and Definition 1.9 of the same root, as well as the transitive rule of the same root in 1.9.1, the conclusion is established.

Proof: Based on Lemma 2.3 and Definition 1.9 of the same root, as well as the transitive rule of the same root in 1.9.1, the conclusion is established.

(4) Corollary: For any class $4k+3, k\in\mathbb{N}$ of positive integer odd numbers x_0, there exists a smallest positive integer $n_0\in\mathbb{Z}^+$ such that $f^{n_0}(x_0)=12m+5, m\in\mathbb{N}$. Therefore, $f^{n_0-1}(x_0)=8m+3, m\in\mathbb{N}$ must hold.

Proof: When $f^{n_0}(x_0)=12m+5, m\in\mathbb{N}$, utilizing the Collatz inverse mapping to transform $\frac{(12m+5)\times 2-1}{3}=8m+3$ leads to the conclusion $f^{n_0-1}(x_0)=8m+3, m\in\mathbb{N}$ being established.

4. Algorithm

4.1 Algorithm 1: Constructing Interconnected Root Sequences Based on Original Symbols

For any given positive odd integer a, taking a as the original element, we use the Collatz mapping transformation formula $b=\frac{(3\times a)+1}{2^k}$, $k\in\mathbb{Z}^+$ to obtain a unique image b. Then, using the Collatz inverse mapping transformation formula $x_0=\frac{(b\times 2^n)-1}{3}$, let n take the smallest positive integer m that makes the equation hold, and $m\in\mathbb{Z}^+$ obtains the smallest unique original element $x_0=\frac{3a-2^{k-m}+1}{3*2^{k-m}}$. Let the sequence x_n be $x_0\cup x_0\times 4^n+4^{n-1}+...+4^1+1$, $n\in\mathbb{Z}^+$, then the sequence x_n is the interconnected root sequence.

Reliability of the algorithm:

Proof: Clearly, the odd integer x_0 Y b. Let x_j be any term $1\le j\le n$ in the sequence x_n. Since:

$$3\times x_0+1=b\times 2^m, m\in\mathbb{Z}^+$$

then: $3\times x_j+1=(3\times x_0+1)\times 4^j=b\times 2^m\times 4^j$, $j\in\mathbb{Z}^+$

Therefore, x_j Y b is reliable.

Completeness of the algorithm: We prove this using proof by contradiction as follows.

Assuming there exists a positive odd integer k_0 that does not belong to the interconnected root

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sequence x_n, but shares a root with the odd integer b. From the above, it can be concluded that the odd integer x_0 is the Collatz inverse mapping transformation formula $x_0 = \frac{(b \times 2^n) - 1}{3}$, and n is chosen to make the equality hold at the smallest positive integer m. $m \in \mathbb{Z}^+$ obtains the smallest unique odd integer, and x_j is any term in the sequence x_n. According to the Collatz transformation rules, we have: $1 \le j \le n$.

$$3 \times x_j + 1 = (3 \times x_0 + 1) \times 4^j, \quad j \in \mathbb{Z}^+ \tag{1}$$

$$3 \times x_0 + 1 = b \times 2^m, \quad m \in \mathbb{Z}^+ \tag{2}$$

$$3 \times k_0 + 1 = b \times 2^r, \quad r \in \mathbb{Z}^+ \tag{3}$$

$$r > m \tag{4}$$

From equation (1), we obtain the equivalent transformation:

$$\frac{3 \times x_j + 1}{3 \times x_0 + 1} = 4^j \tag{5}$$

Equation (3) subtracting equation (2):

$$3 \times (k_0 - x_0) = b \times 2^m \times (2^{r-m} - 1) \tag{6}$$

Equation (3) divided by equation (2):

$$\frac{3 \times k_0 + 1}{3 \times x_0 + 1} = 2^{r-m} \tag{7}$$

For equation (6), since b is not a branch number, if the equality holds, then $2^{r-m} - 1$ must be divisible by 3.

Thus, $r - m = 2k$, $k \in Z^+$ are gotten, substituting in Equation (7), and reorganizing Equation (7):

$$\frac{3 \times k_0 + 1}{3 \times x_0 + 1} = 4^k \tag{8}$$

Therefore, the odd integer k_0 is the k-th term of sequence x_n, shares the same root with the odd integer b, and contradicts the hypothesis, thus establishing completeness.

4.1.1 Uniqueness of Interrelated Same-Root Sequences

Based on the uniqueness of the Collatz transformation rules, the interrelated same-root sequence x_n is unique.

4.1.2 Purpose and Significance

To effectively and reasonably categorize the disordered positive odd integers according to the Collatz transformation rules, ensuring that the odd integer sequences at the same level share the same transformation path.

For example, for $a = 11$, the Collatz transformation yields 17, and by taking the smallest positive integer 1 for the inverse mapping, we obtain 11. Thus, we can derive the interrelated same-root sequence x_n

$$11 \cup \{11 \times 4^n + 4^{n-1} + \ldots + 4^1 + 1\}, \quad n \in \mathbb{Z}^+.$$

For example, for $a = 53$, the Collatz transformation yields 5, and by taking the smallest positive integer 1 for the inverse mapping, we obtain 3. Thus, we can derive the interrelated same-root sequence.

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$$x_n : 3 \cup \{3 \times 4^n + 4^{n-1} + ... + 4^1 + 1\},\ n \in \mathbb{Z}^+ .$$

4.1.3 General Term Formula and Sequence Matrix for Type $4k+1$ and $k \in \mathbb{N}$ Positive Odd Integers Interrelated Same-Root Sequences.

① Construct the interrelated same-root sequence matrix of type $4k+1$ and $k \in \mathbb{N}$ positive odd integers using the constructive method, starting from the smallest positive odd integer, 1, and sequentially constructing the interrelated same-root sequences according to the 3.1 algorithm.

② Remove all constructed interrelated same-root sequences of $4k+3$ and $k \in \mathbb{N}$ class positive odd integers.

③ Remove duplicates by keeping one instance of all constructed interrelated same-root sequences and organizing them.

④ From the steps of constructing the above interrelated same-root sequences, it can be concluded that all odd integers in this array are initially generated by traversing all positive odd integers and then removing the $4k+3$ and $k \in \mathbb{N}$ class positive odd integers. Therefore, the union of all interrelated same-root sequences corresponds one-to-one with the $4k+1$ and $k \in \mathbb{N}$ class positive odd integers, and the interrelated same-root sequences do not intersect, with an empty intersection.

The general term formula is:

$$\begin{cases} (16m+1) \cup (16m+1) * 4^n + 4^{n-1} + \cdots + 4^1 + 1 \\ (16m+9) \cup (16m+9) * 4^n + 4^{n-1} + \cdots + 4^1 + 1 \qquad m \in \mathbb{N}, n \in \mathbb{Z}^+ \\ (16m+13) \cup (16m+13) * 4^n + 4^{n-1} + \cdots + 4^1 + 1 \end{cases} \tag{9}$$

The interrelated same-root sequence matrix of type n rows, $\times m$ columns, and $4k+1$, $k \in \mathbb{N}$ positive odd integers is as follows:

$$\begin{bmatrix} 1 & 5 & 21 & 85 & 341 & 1365 & 5461 & \cdots \\ 9 & 37 & 149 & 597 & 2389 & 9557 & 38229 & \cdots \\ 13 & 53 & 213 & 853 & 3413 & 13653 & 54613 & \cdots \\ 17 & 69 & 277 & 1109 & 4437 & 17749 & 70997 & \cdots \\ 25 & 101 & 405 & 1621 & 6485 & 25911 & 103765 & \cdots \\ 29 & 117 & 469 & 1877 & 7509 & 30037 & 120149 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

From the multiplicity of integers, the characteristics of the row and column elements are as follows:

Row Characteristics:

(1) The first row consists of all positive odd integers that share the same root with 1.

(2) Each row contains all positive odd integers as interrelated same-root sequences.

(3) The positive odd integers in each row are classified as types $12m+1$, $12m+9$, and $12m+5$ odd integers, as well as $m \in \mathbb{N}$.

It can be seen that different classified odd integers are distributed on the interrelated same-root sequence, sharing the same transformation path.

Column Characteristics:

(4) The first column is. $16m+1$, $16m+9$, $16m+13$ class positive odd integers, $m \in \mathbb{N}$;

(5) The second column and all subsequent columns are $16m+5$ class positive odd integers, $m \in \mathbb{N}$.

Row and Column Characteristics:

From the perspective of the row and column characteristics of the above square matrix, the positive odd integers $4k+1$ and $k \in \mathbb{N}$ reflect the features of the Collatz transformation sequence from different classification angles based on the multiplicity of integers, specifically combining interrelated same-root and directly related same-root sequences.

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4.2 Algorithm 2: Directly Related Same Root Based on Pre-image Inverse Mapping Transformation

Let a be any element (non-leaf number) in the interrelated same-root sequence A_n. Using a as an image and applying the Collatz inverse mapping transformation formula $x_0 = \frac{a \times 2^n - 1}{3}$, take the smallest positive integers k and $k \in \mathbb{Z}^+$ that satisfy the equation. Obtain the smallest positive odd integer x_0 and let the sequence x_n be $x_0 \cup x_0 \times 4^n + 4^{n-1} + ... + 4^1 + 1$ and $n \in \mathbb{Z}^+$. It is clear that sequence x_n and sequence A_n share the same root.

Proof: The conclusion can be established similarly to the proof process of Algorithm 3.1.

4.2.1 Purpose and Significance

To connect the originally distinct interrelated same-root sequences, allowing them to share the same transformation path.

For example, interrelated same-root sequences. A_n: $17 \cup \{17 \times 4^n + 4^{n-1} + ... + 4^1 + 1\}$, $n \in \mathbb{Z}^+$.

Choose any odd number from the sequence, such as 17, and take the smallest positive integer, which is 1. According to Algorithm 3.2, the inverse mapping transformation yields the pre-image 11, thus obtaining the sequence. x_n: $11 \cup \{11 \times 4^n + 4^{n-1} + ... + 4^1 + 1\}$, $n \in \mathbb{Z}^+$.

Directly related to sequence A_n through the same root.

4.2.2 Image and Original Inverse Mapping Transformation Same Root Model

Using the odd numbers of type $12m+5$ in the interleaved same root sequence as the image, based on the three transformation path models of the Collatz conjecture in section 3.3.2 and the rules of the Collatz inverse mapping transformation in Algorithm 4.2, they can be categorized into the following four types of inverse mapping transformation models.

(1) Odd numbers of type $12k+3$ share the same root, based on the inverse mapping transformation algorithm in Algorithm 4.2, which involves multiplying by 2, subtracting 1, and dividing by 3.

The traversal period of a complete Collatz inverse mapping transformation (see Figure 4): $12m+5 \to 12k+3$

There is only one type corresponding to the same root sequence: $16k+13, k \in \mathbb{N}$

According to Section 3.2, the general term formula for the same root sequence is:

$$16m+13 \cup \{(16m+13) \times 4^n + 4^{n-1} + ... + 4^1 + 1\}, \quad m \in \mathbb{N}, n \in \mathbb{Z}^+$$

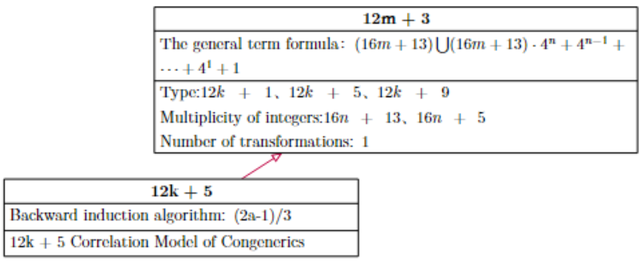

Figure 4: Collatz Inverse Mapping Transformation Same Root Connection Model

For example, for the odd number of type $12m+5$, 5, according to the inverse mapping transformation algorithm in Algorithm 3.2, multiplying by 2, subtracting 1, and dividing by 3 yields the odd number 3.Thus, the $4k+1$ type odd number same root sequence is {13, 53, ...} (excluding the type k odd number 3).

(2) Odd numbers of types $12k+7$, $12k+11$, and $16k+9$ are of the same root, based on the inverse

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mapping transformation algorithm in Algorithm 3.2, which involves multiplying by 2, subtracting 1, dividing by 3, and multiplying by 4, then subtracting 1, and dividing by 3.

The traversal period of a complete Collatz inverse mapping transformation (see Figure 5):

$$12m+5 \rightarrow \underbrace{12k+11...2k+11}_{finitely\ \text{repeated}} \rightarrow 12k+7 \rightarrow 12k+9$$

There are only two types corresponding to the same root sequence: $16k+9, 16k+13, k \in \mathbb{N}$

According to 4.2, the general term formula for the same root sequence is:

$$16m+13 \cup \{(16m+13)\times 4^n + 4^{n-1} + ... + 4^1 + 1\}, \quad m \in \mathbb{N}, n \in \mathbb{Z}^+$$

$$16k+9 \cup \{(16k+9)\times 4^n + 4^{n-1} + ... + 4^1 + 1\}, \quad k \in \mathbb{N}, n \in \mathbb{Z}^+$$

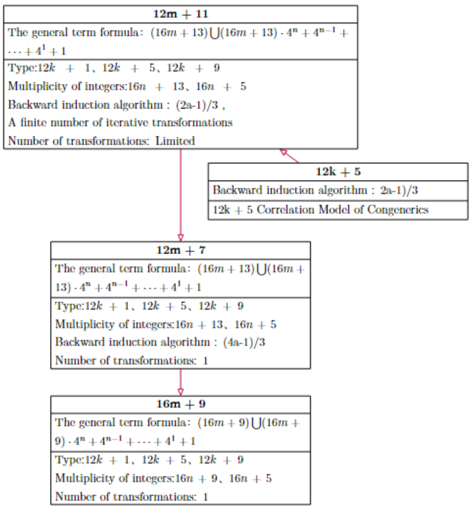

Figure 5: Collatz Inverse Mapping Same Root Connection Model

For example, for the odd number of type $12m+5$, 17, according to the inverse mapping transformation algorithm in Algorithm 3.2, multiplying by 2, subtracting 1, and dividing by 3 sequentially yields the odd numbers 11 and 7. Multiplying by 4 and subtracting 1, then dividing by 3, yields the odd number 9.

Thus, the $4k+1$ type same root sequences are {45, 181, ...}, {29, 117, ...}, and {9, 37, ...} (excluding the type $4k+3$ odd numbers 11 and 7).

(3) Odd numbers of type $12k+7$ and type $16k+9$ are of the same root, based on the inverse mapping transformation algorithm in Algorithm 3.2, which involves multiplying by 2, subtracting 1, dividing by 3, and multiplying by 4, then subtracting 1, and dividing by 3.

The traversal period of a complete Collatz inverse mapping transformation (see Figure 6): $12m+5 \rightarrow 12k+7 \rightarrow 12k+9$

There are only two types corresponding to the same root sequence: $16k+9, 16k+13, k \in \mathbb{N}$

According to 4.2, the general term formula for the same root sequence is:

$$16m+13 \cup \{(16m+13)\times 4^n + 4^{n-1} + ... + 4^1 + 1\}, \quad m \in \mathbb{N}, n \in \mathbb{Z}^+$$

$$16k+9 \cup \{(16k+9)\times 4^n + 4^{n-1} + ... + 4^1 + 1\}, \quad k \in \mathbb{N}, n \in \mathbb{Z}^+$$

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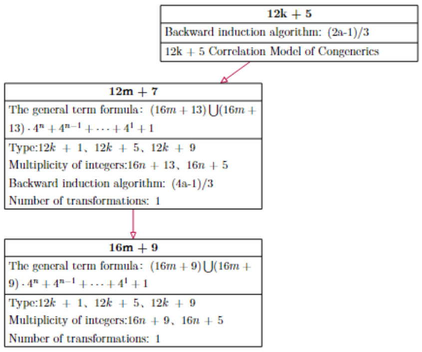

Figure 6: Collatz Inverse Mapping Transformation Same Root Connection Model

For example: For the odd number of type $12m+5$ which is 29, based on Algorithm 3.2, the inverse mapping transformation algorithm involves multiplying by 2, subtracting 1, and dividing by 3 to obtain the odd number 19. Similarly, multiplying by 4, subtracting 1, and dividing by 3 yields the odd number 25.

Thus, the odd root sequences of type $4k+1$ are {77,309,….} and {25,101,….} (with the odd number of type $4k+3$ which is 19 excluded).

(4) For odd numbers of types $12k+3$ and $12k+11$ that share the same root, based on Algorithm 3.2, the inverse mapping transformation algorithm involves multiplying by 2, subtracting 1, and dividing by 3 (see Figure 7).

A complete Collatz inverse mapping transformation traversal cycle:

$$12m+5 \rightarrow \underbrace{12k+11...2k+11}_{\textit{finitely}\ \text{repeated}} \rightarrow 12k+3$$

There is only one type corresponding to the same root sequence: $16k+13, k \in \mathbb{N}$

Based on 4.2, the general term formula for the same root sequence is:

$$16m+13 \cup \left\{(16m+13)\times 4^n + 4^{n-1} + ... + 4^1 + 1\right\}, \quad m \in \mathbb{N}, n \in \mathbb{Z}^+$$

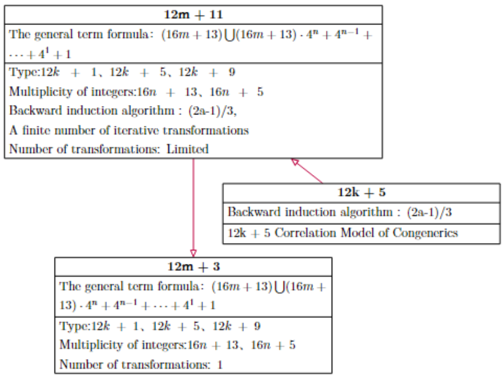

Figure 7: Collatz Inverse Mapping Same Root Connection Model

For example: For the odd number of type $12m+5$, which is 53, based on Algorithm 4.2, the inverse mapping transformation algorithm involves multiplying by 2, subtracting 1, and dividing by 3 in succession to obtain the odd numbers 35, 23, and 15. Thus, the odd root sequences of type $4k+1$ are {141,565,….}, {93,373,….}, and {41,165,….} (with the odd numbers of type $4k+3$, which are 35, 23, and 15, excluded).

4.2.3 For interlinked same-root sequences of types $12m+1$ and $m \in \mathbb{Z}^+$, based on Algorithm 3.2, an associated direct-link same-root model is obtained

For odd numbers of type $16k+1$ that share the same root, based on Algorithm 3.2, the inverse mapping transformation algorithm involves multiplying by 4, subtracting 1, and dividing by 3.

A complete traversal cycle of the Collatz inverse mapping transformation (see Figure 8): $12m+1 \rightarrow 16m+1$

There is only one type corresponding to the same root sequence: $16k+1, k \in \mathbb{N}$

The general term formula for the same root sequence is obtained based on 4.2.:

$$16m+1 \cup \left\{(16m+1)\times 4^{n}+4^{n-1}+\ldots+4^{1}+1\right\}, \quad m \in \mathbb{Z}^{+}, n \in \mathbb{Z}^{+}$$

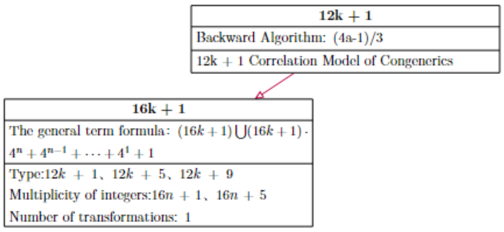

Figure 8: Collatz Inverse Mapping Same-Root Connection Model

For example: For the odd number of type $12m+1$, which is 13, the inverse mapping algorithm involves multiplying by 4, subtracting 1, and dividing by 3 to obtain the odd number 17.

Thus, the same-root sequence of type $4k+1$ is {17,69,….}.

4.2.4 Completeness

Based on the completeness of the inverse mapping transformation in Algorithm 4.2, for any given class $12m+1$ odd number or class $12m+5$ odd number, the indirectly associated same root sequence obtained through the inverse mapping transformation of the given image according to Algorithm 4.2 is complete, with no omissions.

4.2.5 Characteristics of the Same-root Connection Model of Image and Original Object in Inverse Mapping Transformation

Consider the interrelated same-root sequence matrix of types $4k+1$ and $k \in \mathbb{N}$ constructed using the algorithm from 4.1. According to the algorithm in 4.2, for the first column of odd numbers $16m+1$ and $m \in \mathbb{N}$ in the matrix, the distribution of odd numbers in that row is classified into positive integer odd numbers of types $12m+1$, $12m+9$, $12m+5$, and $m \in \mathbb{N}$ For the initial value of odd numbers of type $12m+5$, based on the inverse mapping transformation same-root model in 4.2.2, there exists a one-to-many interconnected same-root sequence of type $16m+9$, $16m+13$, or $16m+9$ with the odd number sequence of type $16m+13$ sharing the same root. For the initial value of odd numbers of type $12m+1$, according to the inverse mapping transformation same-root model in 4.2.3, there exists a one-to-one interconnected same-root sequence of odd numbers of type $16m+1$ sharing the same root.

Similarly, for the first column of odd numbers $16m+9$ and $m \in \mathbb{N}$, the distribution of odd numbers in that row is classified into positive integer odd numbers of types $12m+1$, $12m+9$, $12m+5$, and $m \in \mathbb{N}$. For the initial value of odd numbers of type $12m+5$, according to the inverse mapping transformation same-root model in 4.2.2, there exists a one-to-many interconnected same-root sequence of type $16m+9$, $16m+13$, or $16m+9$ with the odd number sequence of type $16m+13$ sharing the same root. For the initial value of odd numbers of type $12m+1$, based on the inverse mapping transformation same-root model in 4.2.3, there exists a one-to-one interconnected same-root sequence of odd numbers of type $16m+1$sharing the same root.

Similarly, for the first column of odd numbers $16m+13$ and $m \in \mathbb{N}$, the distribution of odd numbers in that row is classified into positive integer odd numbers of types $12m+1$, $12m+9$, $12m+5$, and $m \in \mathbb{N}$. For the initial value of odd numbers of type $12m+5$, according to the inverse mapping transformation same-root model in 4.2.2, there exists a one-to-many interconnected same-root sequence

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of type $16m+9$, $16m+13$, or $16m+9$ with the odd number sequence of type $16m+13$ sharing the same root. For the initial value of odd numbers of type $12m+1$, based on the inverse mapping transformation same-root model in 4.2.3, there exists a one-to-one interconnected same-root sequence of the odd number sequence of type $16m+1$ sharing the same root.

4.2.6 Same-root Connection Method of Image and Original Object in Inverse Mapping Transformation

(1) Existence: $16k+1\overset{\overleftarrow{C}}{\underset{12m+5}{\text{———}}}16m+13$

(2) Existence: $16k+13\overset{\overleftarrow{C}}{\underset{12m+5}{\text{———}}}16m+9$, $16k+9\overset{\overleftarrow{C}}{\underset{12m+5}{\text{———}}}16m+13$

(3) Non-existence: $16k+9\overset{\overleftarrow{C}}{\underset{12m+5}{\text{———}}}16m+1$, $16k+13\overset{\overleftarrow{C}}{\underset{12m+5}{\text{———}}}16m+1$

(4) Non-existence: $16k+9\overset{\overleftarrow{C}}{\underset{12m+1}{\text{———}}}16m+1$, $16k+13\overset{\overleftarrow{C}}{\underset{12m+1}{\text{———}}}16m+1$

(5) Existence: $16k+1\overset{\overleftarrow{C}}{\underset{12m+1}{\text{———}}}16m+1$

(6) Non-existence: $16k+1\overset{\overleftarrow{C}}{\underset{12m+1}{\text{———}}}16m+9$, $16k+1\overset{\overleftarrow{C}}{\underset{12m+1}{\text{———}}}16m+13$

Proof: Based on Lemma 4.2.5 and Definition 2.8, it can be concluded that the proposition holds true.

4.2.7 Equivalent Proposition: For any sequence of indirectly associated same root odd integers of class $16k+1$ that are positive integers, if the Collatz conjecture holds true for 1, then the Collatz conjecture must necessarily hold for any class of positive odd integers

Proof: For any sequence of positive odd integers of class $16m+9$, $16m+13$, or $m \in \mathbb{N}$, based on Lemma 4.2.5, it can be established that for any odd integer $12m+1$ or $m \in \mathbb{N}$ from the sequence, there exists a one-to-one correspondence to the original integers $16k+1$ and $k \in \mathbb{N}$ that also share the same root. Therefore, the equivalent proposition holds true.

5. Lemma

(1) Lemma: The root nodes of a matrix of n rows and m columns consisting of odd positive integer sequences of types $4k+1$ and $m \in \mathbb{N}$ are odd positive integers of types $12m+5$ and $m \in \mathbb{N}$.

Proof: According to Corollary 3.3.3, for any odd positive integers of types $4k+3$ and $k \in \mathbb{N}$, there exists a unique image of odd positive integers of types $12m+5$ and $m \in \mathbb{N}$ corresponding to their Collatz mapping transformation, and the mapping relationship is surjective. Conversely, based on Algorithm 4.2, using odd positive integers of type $12m+5$ as images to perform a complete traversal of the Collatz inverse mapping transformation yields all the corresponding pre-images of odd positive integers of type $4k+3$; the inverse mapping relationship is one-to-many, and the resulting odd positive integers of types $4k+3$ and $k \in \mathbb{N}$ are complete, with no omissions. Therefore, the root nodes are odd positive integers of types $12m+5$ and $m \in \mathbb{N}$. Thus, the proposition holds.

(2) Lemma: For a matrix of n rows and m columns consisting of odd positive integer sequences of type $4k+1$, with all odd numbers of types $16m+9$, $16m+13$, and $m \in \mathbb{N}$ as root nodes, and odd numbers of types $12m+1$ and $m \in \mathbb{N}$ on the root node sequence as images, a complete Collatz inverse mapping transformation can be performed, traversing the same-root model according to 4.2.3. Based on the characteristics of the same-root model in the inverse mapping transformation outlined in 4.2.5, it can

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be concluded that the odd sequences of types $16m+1$ and $m \in \mathbb{N}$ that meet this condition will ultimately be merged into the odd root node sequences that share the same root.

Similarly, by using all odd sequences of types $12m+1$ and $m \in \mathbb{N}$ on the obtained same-root sequences of odd types $16m+1$ and $m \in \mathbb{N}$ as images, a complete Collatz inverse mapping transformation can be performed, traversing the same-root model according to the inverse mapping transformation of 4.2.3. Based on the characteristics of the same-root model in the inverse mapping transformation outlined in 4.2.5, the resulting odd sequences of types $16m+1$ and $m \in \mathbb{N}$ that meet this condition can be merged into the odd root node sequences that share the same root.

By extension, the following conclusion can be drawn regarding the same-root sequences constructed with odd sequences of types $16m+9$, $16m+13$, and $m \in \mathbb{N}$ as root nodes:

① Distinct root node sequences do not intersect with each other;

② The same root node sequence does not produce cycles;

③ In a matrix of n rows and m columns, there exist odd sequences of types $16m+1$ and $m \in \mathbb{N}$ that have not been merged into the odd root node sequences of types $16m+9$ and $16m+13$ that share the same root.

Proof: ① Using proof by contradiction, assume that there exists a root node sequence that intersects with another root node sequence. Conversely, taking the intersecting odd number as the initial value for the Collatz transformation would result in the 2.5 Collatz transformation path, violating the uniqueness of the Collatz transformation. Therefore, the original proposition holds.

② Since the root nodes are odd numbers of types $16k+9$, $16k+13$, and $k \in \mathbb{N}$, the root node sequences consist of odd positive integers of types $16k+9$, $16k+13$, $16k+5$, and $k \in \mathbb{N}$. By using constructive inverse mapping transformations, the merged same-root odd sequences are all of types $16m+1$ and $m \in \mathbb{N}$; therefore, the odd sequences merged through this constructive method cannot produce cycles with odd sequences of types $16k+9$, $16k+13$, $16k+5$, and $k \in \mathbb{N}$. On the other hand, if there exist odd numbers of types $16m+1$ and $m \in \mathbb{N}$ that generate cycles between each other, then there would be a cyclical odd number as the initial value for the Collatz transformation, which would result in the 2.5 Collatz transformation path, violating the uniqueness of the Collatz transformation path. Thus, the original proposition holds.

③ The first row of the 1-cycle sequence and the sequences that share the same root as the 1-cycle sequence have not been merged, thus the proposition holds. Using proof by contradiction, we can demonstrate that if there exists a sequence sharing the same root as the 1-cycle sequence that has been merged, then the Collatz mapping transformation for that sequence would not be unique, which contradicts the uniqueness of the Collatz mapping transformation.

(3) Lemma: The root nodes of a matrix of odd positive integer sequences consisting of types n and $m \in \mathbb{N}$ with n rows and m columns are odd positive integers of types $16m+1$ and $m \in \mathbb{N}$.

Proof: According to Lemma 5(1), it is known that the root nodes are odd positive integers of types $12m+5$ and $m \in \mathbb{N}$. Since odd positive integers of types $12m+5$ and $m \in \mathbb{N}$ are distributed among the odd positive integer sequences of types $16k+1$, $16k+9$, $16k+13$, and $k \in \mathbb{N}$. According to Lemma 5(2), if odd numbers of types $16k+9$, $16k+13$, and $k \in \mathbb{N}$ are taken as root nodes, with odd numbers of types $12m+1$ and $m \in \mathbb{N}$ on the root node sequences as images, the odd positive integer sequences of types $16m+1$ and $m \in \mathbb{N}$ obtained via the inverse mapping transformation 4.2.3 are incomplete. Therefore, odd number sequences of types $16k+9$, $16k+13$, and $k \in \mathbb{N}$ cannot serve as root node sequences, leading to the conclusion that the original proposition holds. On the other hand, from the inverse mapping relations of 4.2.6(1), 4.2.6(2), and 4.2.6(3), it can be concluded that odd positive integer sequences of type $16m+1$, $m \in \mathbb{N}$ serve as root nodes.

(4) Lemma: For a matrix of odd number sequences consisting of positive integers of types $4k+1$ and $k \in \mathbb{N}$ with n rows and m columns, according to Lemma 5(3), taking all odd number sequences of types $16k+1$ and $k \in \mathbb{N}$ as root nodes, and all odd numbers of types $12m+5$ and $m \in \mathbb{N}$ on the

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odd number sequences of type $16k+1$ as images, a complete Collatz inverse mapping transformation cycle traversal is performed based on the same-root model of inverse mapping transformation 4.2.2. According to the characteristics of the same-root model of inverse mapping transformation 4.2.5, the obtained odd number sequences of types $16m+9$, $16m+13$, and $m \in \mathbb{N}$ will be merged with the root node odd number sequences of types $16k+1$ and $k \in \mathbb{N}$.

Similarly, the odd number sequences of types $16m+9$, $16m+13$, and $m \in \mathbb{N}$ obtained, with all odd numbers of type $12m+5$ as images, undergo a complete Collatz inverse mapping transformation cycle traversal based on the same-root model of inverse mapping transformation 4.2.2. According to the characteristics of the same-root model of inverse mapping transformation 4.2.5, the obtained odd number sequences of types $16m+9$, $16m+13$, and $m \in \mathbb{N}$ will be merged with the root node sequences of types $16k+1$ and $k \in \mathbb{N}$.

By this analogy, the following conclusions can be drawn regarding the same-root sequences constructed through inverse mapping with odd number sequences of types $16k+1$ and $k \in \mathbb{N}$ as root nodes:

① Distinct root node sequences do not intersect;

② The same root node sequence does not produce cycles.

③In the matrix with n rows and m columns, all odd number sequences of types $16m+9$, $16m+13$, and $m \in \mathbb{N}$ are ultimately merged into the same-root odd number sequences of types $16k+1$ and $k \in \mathbb{N}$, with no omissions.

Proof: ① Based on the proof process of the conclusion in Lemma 5(2)①, a similar argument can be used to prove that the proposition holds true.

② Since the root nodes are odd numbers of types $16k+1$ and $k \in \mathbb{N}$, the sequences above them consist of odd numbers of types $16k+1$, $16k+5$, and $k \in \mathbb{N}$. Through the construction method of inverse mapping transformations, the merged same-root odd number sequences are of types $16m+9$ or $16m+13$ and $m \in \mathbb{N}$. Therefore, the odd number sequences merged through the construction method will not produce cycles with odd numbers of types $16k+1$, $16k+5$, or $k \in \mathbb{N}$. On the other hand, if a cycle were to occur between odd numbers of types $16m+9$ or $16m+13$ and $m \in \mathbb{N}$, it would imply that the odd number at the cycle point is used as the initial value for the Collatz mapping transformation, leading to a Collatz transformation path as stated in 2.5. This would violate the uniqueness of the Collatz transformation path. Thus, the original proposition holds true.

③According to Lemma 5(3), it is evident that the positive integer odd numbers of types $16m+1$ and $m \in \mathbb{N}$ serve as root node sequences. Therefore, the same-root sequences of types $16m+9$, $16m+13$, and $m \in \mathbb{N}$ constructed from the root node sequences are complete, with no omissions. On the other hand, based on the completeness of algorithms 4.1 and 4.2, the characteristic initial values of the interleaved same-root sequences constructed on all root node sequences using algorithm 4.1, which are odd numbers of types $12m+5$ and $m \in \mathbb{N}$, are complete. Furthermore, for any given positive integer odd numbers of types $12m+5$ and $m \in \mathbb{N}$, the interleaved same-root sequences obtained through the Collatz inverse mapping transformation traversal of the given image's completeness using algorithm 4.2 are also complete. Thus, the conclusion of completeness holds.

(5) Lemma: The Collatz conjecture transformation of any positive integer odd numbers of types $4k+1$ and $k \in \mathbb{N}$ must be 1.

Proof: Using the construction method and mathematical induction.

①In the first step, the construction method is used. For the matrix of positive integer odd number sequences of type $4k+1$ in n rows and m columns, the same-root sequences of root nodes are constructed using all odd number sequences of types $16k+1$ and $k \in \mathbb{N}$, based on Lemma 5(4).

It can be seen that after the first step of construction, each root node sequence of types $16k+1$ and

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$k \in \mathbb{N}$ has infinitely many odd number sequences of types $16m+9$ or $16m+13$ and $m \in \mathbb{N}$ sharing the same root. Conversely, any odd number on the same-root related sequences of types $16m+9$, $16m+13$, and $m \in \mathbb{N}$, after a finite number of Collatz transformations, will have the same transformation path as the odd numbers of types $12m+5$ and $m \in \mathbb{N}$ on the root node sequences of types $16k+1$ and $k \in \mathbb{N}$. The matrix of root node sequences after the first step of construction and merging is as follows:

$$\begin{bmatrix} 1 & 5 & 21 & 85 & 341 & 1365 & 5461 & \cdots \\ 17 & 69 & 277 & 1109 & 4437 & 17749 & 70997 & \cdots \\ 33 & 133 & 533 & 2133 & 8533 & 34133 & 136533 & \cdots \\ 49 & 197 & 789 & 3157 & 12629 & 50517 & 202069 & \cdots \\ 65 & 261 & 1045 & 4181 & 16725 & 66901 & 267605 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

②The second step continues to employ the construction method. Using all odd number sequences of types $16m+9$ or $16m+13$ and $m \in \mathbb{N}$ merged in the first step as root nodes, the same-root sequences are constructed based on Lemma 5(2).

Based on the multiplicity of integers, odd numbers of types $12k+1$ and $k \in \mathbb{N}$ can be divided into odd numbers of types $16k+1$, $16k+5$, $16k+9$, $16k+13$, and $k \in \mathbb{N}$. After the second step of construction, the merged sequences of odd numbers, $16k+1$ and $k \in \mathbb{N}$, undergo Collatz transformations resulting in odd numbers of types $12k+1$ and $k \in \mathbb{N}$ The odd numbers $12k+1$ and $k \in \mathbb{N}$ are represented as same-root sequences of odd numbers $16k+9$, or $16k+13$ and $k \in \mathbb{N}$, which are merged by the root node sequence. Therefore, for all unmerged root node odd number sequences that underwent the second-step same-root construction, their Collatz mapping transformation (resulting in odd numbers of types $12k+1$ and $k \in \mathbb{N}$) must be odd number sequences of types $16m+1$ and $m \in \mathbb{N}$ that are less than themselves. Thus, the matrix of the root node sequences after the second step of the construction and merging is as follows:

$$\begin{bmatrix} 1 & 5 & 21 & 85 & 341 & 1365 & 5461 & \cdots \\ 113 & 453 & 1813 & 7253 & 29013 & 116053 & 464213 & \cdots \\ 2417 & 9669 & 38677 & 154709 & 618837 & 2475349 & 9901397 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

③To prove the unmerged odd sequences of types $16k+1$ and $k \in \mathbb{N}$ after the second step of the same root construction using mathematical induction.

①When $n=1$ is considered, the first row of odd sequences is rooted in 1, and thus the proposition holds true.

②Assuming that when $n<k$ is in consideration, the proposition holds true. That is, after the second step of the same root construction, the unmerged odd sequences of types $16k+1$ and $k \in \mathbb{N}$ in the k -th row are rooted in 1.

③When $n=k$ is considered, it can be inferred from the above that after the second step of the same root construction, the k -th row of odd sequences $16k+1$ and $k \in \mathbb{N}$ that have not been merged will, after a finite number of Collatz transformations, become odd sequences of types $16m+1$ and $m \in \mathbb{N}$ that are less than themselves. Therefore, the k -th row $16k+1$ of odd sequences is rooted in 1. In summary, the original proposition holds true.

6. Conclusion

The application of constructive methods and mathematical induction is extensive; both methods are indispensable proof tools in both elementary and advanced mathematics. This paper has conducted an in-depth and detailed study and analysis of the Collatz conjecture. By introducing the concept of same

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roots and employing the idea of constructing before proving, we have utilized constructive methods and mathematical induction to demonstrate that the conjecture holds true for any odd positive integer greater than 1.

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