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Research on the Existence Criterion for Subgroup Perfect Codes in Bi-Cayley Graphs

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Abstract: *In a graph Γ, a perfect code C is an independent set of the vertex set $V(\Gamma)$ such that every vertex outside C is adjacent to exactly one vertex in C. This notion generalizes classical perfect error-correcting codes and has been extensively studied in Cayley graphs. This paper extends the concept of perfect codes in Cayley graphs to Bi-Cayley graphs and presents a criterion for determining the existence of subgroup perfect codes in Bi-Cayley graphs. Let G be a finite group and let H, K≤G. The pair (H,K) is called a subgroup perfect code of (G,G) if there exists a bi-Cayley graph over G such that $H_0 \cup K_1$ forms a perfect code. The main result provides a necessary and sufficient condition for (H,K) to be such a code: there exists a subset $S \subseteq G$ such that H has an inverse-closed transversal in $G \setminus ((S^{-1}*K) \sqcup H)$ and K has an inverse-closed transversal in $G \setminus ((S*H) \sqcup K)$, where S*H denotes the disjoint union of left cosets. The result offers a systematic method to construct and verify perfect codes in bi-Cayley graphs, and reveals how group structure determines combinatorial domination in symmetric graphs.*

Keywords: *perfect code, bi-Cayley graph, subgroup perfect code*

1. Introduction

Perfect code is a fundamental concept in algebraic graph theory, originating from the connection between coding theory and graph structures. A perfect code is an independent set of vertices in a graph such that every vertex outside the set is adjacent to exactly one vertex in the set. Such codes capture both the combinatorial structure and adjacency relations of graphs, and play an important role in coding theory, cryptography, network optimization, and the study of symmetry in groups and graphs.

Cayley graphs are highly symmetric regular graphs constructed from groups, and they serve as a classical setting for perfect codes. The theory of perfect codes in Cayley graphs is now well developed. Bi-Cayley graphs are more general and structurally richer than Cayley graphs, while still preserving strong symmetry and regularity. They provide a broader framework for understanding the intrinsic relationship between groups and graphs. However, research on perfect codes in bi-Cayley graphs remains limited. In particular, there is no direct and efficient criterion to determine whether a subgroup perfect code exists in such graphs, which restricts the generalization of perfect code theory to more general graph classes.

This paper extends the concept of perfect codes from Cayley graphs to bi-Cayley graphs and establishes a criterion for the existence of subgroup perfect codes in this setting. The criterion is explicit and easy to apply, allowing one to verify the existence of such codes directly from the group structure. Theoretically, this work fills a gap in the study of perfect codes beyond Cayley graphs and strengthens the connection among group theory, graph structure, and combinatorial coding. It also offers new tools for analyzing the symmetry and adjacency properties of regular graphs. Moreover, the ideas developed here provide a foundation for further research on coding applications of bi-Cayley graphs, network topology design, and related algebraic structures, contributing to the interdisciplinary development of graph theory and coding theory.

2. Main results

In this paper, all groups are finite, and all graphs are finite, undirected and simple.

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For $C \subseteq V(\Gamma)$, if no two vertices in C are adjacent, then C is called an independent set. For an independent set $C \subseteq V(\Gamma)$, if every vertex not in C is adjacent to exactly one vertex in C, then C is called a perfect code for Γ. In graph theory, perfect codes in graphs are also referred to as *efficient dominating sets* [1] or *independent perfect dominating*

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sets [2].

The notion of perfect codes in graphs was first introduced by Biggs [3] as a generalization of the classical concept of perfect t −error-correcting code in coding theory. In recent years, numerous researchers have devoted their efforts to studying perfect codes in Cayley graphs (see, for example, [4, 5, 6, 7, 8, 9, 10]). As a generalization of Cayley graphs, bi-Cayley graphs have garnered significant attention in both group theory and graph theory (see, for example, [11, 12, 13, 14]). This paper focuses on the study of perfect codes in bi-Cayley graphs.

Let S be an inverse-closed subset of G not containing the identity element 1. The *Cayley graph* $\mathrm{Cay}(G,S)$ of G with a connection set S is defined to be the graph with vertex set G such that two elements x, y are adjacent if and only if $yx^{-1}\in S$. A bi-Cayley graph is a generalization of a Cayley graph, defined as follows: Let R, L and S be subsets of a group G such that $R=R^{-1}$, $L=L^{-1}$ and $1\notin R\cup L$. A Bi-Cayley Graph $\mathrm{BiCay}(G,R,L,S)$ is the graph with a vertex set $G\times\{0,1\}$ and two vertices (h,i), (g,j) are adjacent if and only if one of the following occurs:

(1) $i=j=0$ and $gh^{-1}\in R$;

(2) $i=j=1$ and $gh^{-1}\in L$;

(3) $i=0, j=1$ and $gh^{-1}\in S$.

For any subset H of G, let $H_i=\{(h,i)\mid h\in H\}$ for $i=0$ or 1. In[6], Huang, Xia and Zhou introduced the following concept: A subset H of G is *a perfect code* of G if there exists Cayley graph $\mathrm{Cay}(G,S)$ of G that admits H as a perfect code. We generalize Huang's concept to bi-Cayley graphs as follows:

Definition 2.1 *Let G be a group, $H\subseteq G$ and $K\subseteq G$. If there exists a bi-Cayley graph $BiCay(G,R,L,S)$ of G admitting $H_0\cup K_1$ as a perfect code, then the pair (H,K) is called a perfect code of the pair (G,G). If further $H\le G$ and $K\le G$, then (H,K) is called a subgroup perfect code of (G,G).*

For a subgroup H of G and a subset S, let $S*H=\sqcup_{s\in S}sH$, where $\sqcup$ denotes the disjoint union. This concept will be used implicitly when we refer to inverse-closed transversals in the proof of Theorem 2.2. According to Definition 1, we have provided a characterization for the subgroup perfect codes of (G,G).

Theorem 2.2 *Let G be a group, $H\le G$ and $K\le G$. Then (H,K) is a perfect code of (G,G) if and only if there exists $S\subseteq G$ such that H and K have inverse-closed transversals in $G\setminus((S^{-1}*K)\sqcup H)$ and $G\setminus((S*H)\sqcup K)$, respectively.*

3. Preliminaries

Firstly, we introduce some notations used in this paper. For certain group theory terms not defined here, readers may refer to [15]. Let Γ be a graph. For any $\mu\in V(\Gamma)$, we denote the neighborhood of μ by $\Gamma(\mu)$. For two vertices x and y in the graph, $x\sim y$ indicates that x is adjacent to y. Let G be a group and $H\le G$. For any $g\in G$, we denote gH as the left coset of subgroup H in group G. By selecting elements appropriately, some of these left cosets can serve as partitions of G, denoted as $G=\sqcup_{i=1}^{n}g_iH$, where $g_i\in G$. In this case, the set $\{g_1,g_2,\dots,g_n\}$ is called a left transversal of subgroup H in group G. Similarly, for its right cosets, we can obtain a right transversal of subgroup H in group G by selecting appropriate elements.

Let U be an union of some left cosets of H in G. We can define the transversal of H in U just as the definition of the transversal of H in G. Let $S*H=\sqcup_{s\in S}sH$ for some subset S of G. Hence, $SH=S*H$ if and only if SH is a disjoint union of left cosets of H with transversal S, that is, $SH=\sqcup_{s\in S}sH$. Furthermore, we readily obtain the following property:

Property 3.1 *If $SH=S*H$, then $\sqcup_{s\in S}sH=\sqcup_{h\in H}Sh$.*

Proof. In fact, let $S=\{s_1,s_2,\dots,s_m\}$, $H=\{h_1,h_2,\dots,h_n\}$, where m and n are positive integers. It follows that $SH=\{s_1h_1,\dots,s_1h_n,\dots s_mh_n\}=\sqcup_{i=1}^{n}\{s_1h_i,s_2h_i,\dots,s_mh_i\}=\sqcup_{h\in H}Sh$. Since $SH=\sqcup_{s\in S}sH$, we have that $\sqcup_{s\in S}sH=\sqcup_{h\in H}Sh$.

The equality $\sqcup_{s\in S}sH=\sqcup_{h\in H}Sh$ shows that a disjoint union of left cosets can also be viewed as a disjoint union of right cosets of S. This dual perspective is useful when working with bi-Cayley graphs. In the proof of Theorem 2.2, we will encounter sets of the form Rh and Sh for $h\in H$. The condition $SH=S*H$ guarantees that these sets are pairwise disjoint as h varies, which is essential for verifying the perfect code property. A similar observation applies to $S^{-1}K$ and LK.

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According to the definition of a perfect code, we obtain the following lemma.

Lemma 3.2 *Let* Γ *be a graph and* C *be a subset of* $V(\Gamma)$. C *is a perfect code of* Γ *if and only if* $V(\Gamma)=(\sqcup_{\mu\in C}\Gamma(\mu))\sqcup C$.

Proof. We prove the necessity first. Suppose that C is a perfect code in Γ. Then C is an independent set. For any vertex $x\in V(\Gamma)\setminus C$, there exists a unique vertex $\mu\in C$ such that $x\sim\mu$, that is, $x\in\Gamma(\mu)$. Consequently, $x\in\cup_{\mu\in C}\Gamma(\mu)$, and hence

$$V(\Gamma)\setminus C\subseteq\cup_{\mu\in C}\Gamma(\mu).$$

Therefore,

$$V(\Gamma)=(\cup_{\mu\in C}\Gamma(\mu))\cup C.$$

Since C is independent, we have that

$$V(\Gamma)=(\cup_{\mu\in C}\Gamma(\mu))\sqcup C.$$

Now take any two distinct vertices $\mu,\ \nu\in C$. If $\Gamma(\mu)\cap\Gamma(\nu)\neq\emptyset$, then there exists a vertex $p\in V(\Gamma)$ such that $p\sim\mu$ and $p\sim\nu$, a contradiction to the definition of a perfect code. Hence, $\Gamma(\mu)\cap\Gamma(\nu)=\emptyset$. It follows that

$$\cup_{\mu\in C}\Gamma(\mu)=\sqcup_{\mu\in C}\Gamma(\mu).$$

In summary,

$$V(\Gamma)=(\sqcup_{\mu\in C}\Gamma(\mu))\sqcup C.$$

Now, we prove the sufficiency. Assume that

$$V(\Gamma)=(\sqcup_{\mu\in C}\Gamma(\mu))\sqcup C.$$

Then for any $\mu\in C$, we have $\Gamma(\mu)\cap C=\emptyset$. Hence, C is an independent set. For any vertex $\nu\in V(\Gamma)\setminus C$, there exists a unique $\mu\in C$ such that $\nu\in\Gamma(\mu)$, that is, $\mu\sim\nu$. By definition, C is a perfect code in Γ. This prove the sufficiency.

For convenience, we also use “perfect code” instead of “subgroup perfect code” in the rest of the paper.

4. Proof of Theorem 2.2

Lemma 3.2 characterizes a perfect code as a disjoint union of neighborhoods. In a bi-Cayley graph, these neighborhoods are expressed as unions of left cosets. This turns the problem of verifying a perfect code into a decomposition of the group G. Theorem 2.2 is proved by translating such a decomposition into the existence of certain inverse-closed transversals. With the above notation and results, we are now ready to prove Theorem 2.2.

We prove the necessity first. Suppose that (H,K) is a perfect code of (G,G). By Definition 2.1, there exists a bi-Cayley graph $\Gamma=\mathrm{BiCay}(G,R,L,S)$ such that $H_0\cup K_1$ is a perfect code in Γ. The definition of bi-Cayley graph show that $R=R^{-1},L=L^{-1}$ and $1\notin R\cup L$. According to Lemma 3.2, we deduce that

$$V(\Gamma)=G_0\sqcup G_1=(\sqcup_{\nu\in H_0}\Gamma(\nu))\sqcup(\sqcup_{\mu\in K_1}\Gamma(\mu))\sqcup(H_0\sqcup K_1).$$

For any $\nu=(h,0)\in H_0$ and $\mu=(k,1)\in K_1$, it is clear that $\Gamma(\nu)=(Rh)_0\sqcup(Sh)_1$ and $\Gamma(\mu)=(S^{-1}k)_0\sqcup(Lk)_1$. Then $\sqcup_{\nu\in H_0}\Gamma(\nu)=(RH)_0\sqcup(SH)_1$ and $\sqcup_{\mu\in K_1}\Gamma(\mu)=(S^{-1}K)_0\sqcup(LK)_1$. Since $\cup_{\nu\in H_0}\Gamma(\nu)=\sqcup_{\nu\in H_0}\Gamma(\nu)$ and $\cup_{\mu\in K_1}\Gamma(\mu)=\sqcup_{\mu\in K_1}\Gamma(\mu)$, it yields that $RH=R*H,SH=S*H,S^{-1}K=S^{-1}*K,LK=L*K$. Thus that

$$G_0\sqcup G_1=((R*H)_0\sqcup(S*H)_1)\sqcup((S^{-1}*K)_0\sqcup(L*K)_1)\sqcup(H_0\sqcup K_1).$$

It follows that

$$G=(R*H)\sqcup(S^{-1}*K)\sqcup H=(S*H)\sqcup(L*K)\sqcup K.$$

Recalling that $R=R^{-1},\ L=L^{-1}$, we conclude that

$$G\setminus((S^{-1}*K)\sqcup H)=R*H=R^{-1}*H\ \text{ and }\ G\setminus\big((S*H)\sqcup K\big)=L*K=L^{-1}*K.$$

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Then H and K have inverse-closed transversals in $G \setminus ((S^{-1} * K) \sqcup H)$ and $G \setminus ((S * H) \sqcup K)$, respectively.

Now we prove the sufficiency. Assume that there exists $S \subseteq G$ such that H has inverse-closed transversal R in $G \setminus ((S^{-1} * K) \sqcup H)$ and K has inverse-closed transversal L in $G \setminus ((S * H) \sqcup K)$. Then

$$G = (R * H) \sqcup (S^{-1} * K) \sqcup H = (L * K) \sqcup (S * H) \sqcup K$$

Let $\Gamma = \mathrm{BiCay}(G, R, L, S)$. It follows that

$$V(\Gamma) = G_0 \sqcup G_1 = ((R * H) \sqcup (S^{-1} * K) \sqcup H)_0 \sqcup ((L * K) \sqcup (S * H) \sqcup K)_1$$
$$= ((R * H) \sqcup (S^{-1} * K))_0 \sqcup ((L * K) \sqcup (S * H))_1 \sqcup (H_0 \cup K_1)$$

For any $\mu, \nu \in H_0 \cup K_1$ and $\mu \neq \nu$, we claim that $\Gamma(\mu) \cap \Gamma(\nu) = \emptyset$ and proceed by dividing the argument into following cases.

Case 1. μ and ν are both in H_0.

In this case, let $\mu = (h, 0)$ and $\nu = (h', 0)$ for some $h, h' \in H$. Recalling that $SH = S * H = \sqcup_{s \in S} sH = \sqcup_{h \in H} Sh$, we deduce that $Sh \cap Sh' = \emptyset$. Similarly, $Rh \cap Rh' = \emptyset$. Since

$$\Gamma(u) \cap \Gamma(v) = ((Rh)_0 \sqcup (Sh)_1) \cap ((Rh')_0 \sqcup (Sh')_1) = (Rh \cap Rh')_0 \sqcup (Sh \cap Sh')_1,$$

we conclude that$\Gamma(\mu) \cap \Gamma(\nu) = \emptyset$.

Case 2. μ and ν are both in K_1.

In this case, let $\mu = (k, 1)$ and $\nu = (k', 1)$, for some $k, k' \in K$. Similar to case 1, we deduce that $S^{-1}k \cap S^{-1}k' = \emptyset$ and $Lk \cap Lk' = \emptyset, Lk \cap Lk' = \emptyset$. Since

$$\Gamma(\mu) \cap \Gamma(\nu) = ((S^{-1}k)_0 \sqcup (Lk)_1) \cap ((S^{-1}k')_0 \sqcup (Lk')_1) = (S^{-1}k \cap S^{-1}k')_0 \sqcup (Lk \cap Lk')_1,$$

we conclude that $\Gamma(\mu) \cap \Gamma(\nu) = \emptyset$.

Case 3. μ and ν are not both in H_0 or K_1.

In this case, without loss of any generality, let $\mu = (h, 0)$ and $\nu = (k, 1)$ for some $h \in H, k \in K$. Since $(R * H) \cap (S^{-1} * K) = \emptyset$ and $(S * H) \cap (L * K) = \emptyset$, we deduce that

$$\Gamma(\mu) \cap \Gamma(\nu) = ((Rh)_0 \sqcup (Sh)_1) \cap ((S^{-1}k)_0 \sqcup (Lk)_1) = (Rh \cap S^{-1}k)_0 \sqcup (Sh \cap Lk)_1 = \emptyset.$$

Hence, the claim holds. Now, we conclude that:

$$\sqcup_{x \in H_0 \sqcup K_1} \Gamma(x) = (\sqcup_{\mu \in H_0} \Gamma(\mu)) \sqcup (\sqcup_{\nu \in K_1} \Gamma(\nu)) = ((R * H)_0 \sqcup (S * H)_1) \sqcup ((S^{-1} * K)_0 \sqcup (L * K)_1)$$
$$= ((R * H) \sqcup (S^{-1} * K))_0 \sqcup ((S * H) \sqcup (L * K))_1.$$

Then we have that $V(\Gamma) = \sqcup_{x \in H_0 \cup K_1} \Gamma(x) \sqcup (H_0 \cup K_1)$. According to Lemma 3.2, we conclude that $H_0 \cup K_1$ is a perfect code of Γ. Therefore (H, K) is a perfect code of (G, G).

5. Conclusion

This paper introduced the concept of perfect codes into the setting of bi-Cayley graphs, thereby extending the classical theory of subgroup perfect codes from Cayley graphs to a broader family of regular graphs. We proposed the notion of a perfect code of the pair (G, G) and, more specifically, a subgroup perfect code, in which the code takes the form $H_0 \cup K_1$ with $H, K \leq G$. The main result, Theorem 2.2, provides a necessary and sufficient condition for (H, K) to be such a code: the existence of a $S \subseteq G$ such that H and K admit inverse-closed transversals in the complements of $(S^{-1} * K) \sqcup H$ and $(S * H) \sqcup K$respectively. This criterion not only characterizes subgroup perfect codes in bi-Cayley graphs in a constructive manner, but also recovers the classical Cayley graph case when $H = K$ and the transversal condition is suitably specialized. The result offers a systematic approach to constructing and verifying perfect codes in bi-Cayley graphs, and contributes to a deeper understanding of the interplay between group structure and combinatorial regularity in highly symmetric graphs.

Despite the theoretical completeness of Theorem 2.2, two main limitations remain in the present work. The first is the lack of effective methods to determine, for a given group G and its subgroups H and K, whether a suitable subset S and the required inverse-closed transversals actually exist. The condition

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given in the theorem is clear and group-theoretic, but it does not directly tell us how to find such S and transversals in practice. When G is large or non-abelian, the range of possibilities becomes too large to handle, and no efficient algebraic or combinatorial method has been developed to simplify this checking. The second limitation is that the current work only deals with subgroup codes, that is, both H and K are required to be subgroups of G. The more general case where H and K are arbitrary subsets—not necessarily subgroups—remains untouched. This general case is expected to be much more difficult and will require new ideas beyond coset decompositions and transversals.

These limitations naturally suggest several directions for future research. One immediate task is to apply the criterion to concrete families of finite groups. Cyclic groups and dihedral groups are good candidates. These groups have simple subgroup structures and well-understood coset decompositions, making them suitable for explicit constructions. For such groups, one can try to find all possible triples (H, K, S) that satisfy the transversal condition, and then obtain either explicit perfect codes or rigorous non-existence results. These case studies will not only test the sharpness of the theorem but also provide concrete examples that are useful for further theoretical development.

Another important direction is to extend the framework to other classes of bi-Cayley graphs. This paper focuses on undirected bi-Cayley graphs, where the connection sets R and L are inverse-closed and the edge set is symmetric. A natural generalization is to consider directed bi-Cayley graphs, where R and L are not required to be inverse-closed. In this setting, the notion of perfect codes and the corresponding criterion may take different forms and need further study.

Beyond the scope of finite groups, another possible direction is to consider infinite groups. The current work assumes G is finite, which is essential for the combinatorial counting arguments in the proof. However, bi-Cayley graphs over infinite groups also appear in many contexts. Extending the theory of subgroup perfect codes to infinite groups would require new techniques, particularly in handling infinite coset decompositions and transversals.

In addition, the potential applications of subgroup perfect codes in bi-Cayley graphs are worth exploring. In cryptography, hard problems are widely used to build secure systems. Whether subgroup perfect codes can give rise to a new type of hard problem is completely unknown and requires further investigation.

In communication systems, error-correcting codes with low decoding complexity are highly desirable, especially in environments where delay is critical. Subgroup perfect codes have a natural decoding advantage: by definition, every vertex outside the code is adjacent to exactly one vertex inside the code. This one-to-one correspondence allows for clear and immediate error correction without iterative decoding. Whether this property can be used in practical delay-sensitive applications, such as tactical data links or anti-jamming systems, is an open question that requires joint work between coding theorists and graph theorists.

In summary, the theoretical framework established in this paper provides a solid foundation for further research on perfect codes in bi-Cayley graphs. Closing the gap in computational methods, extending the code family to more general settings, and exploring possible connections with real-world applications are the main tasks ahead. Progress in these directions will not only strengthen the theory but also increase its impact across mathematics and engineering.

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