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Generalized application of differential mean value theorem in infinite intervals

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Abstract: *As one of the core ideas and courses of mathematical analysis, differential calculus plays a pivotal role in the study of functions and is an indispensable part of the composition of mathematical analysis, especially Rolle's Median Theorem, Lagrange's Median Theorem, Cauchy's Median Theorem, and so on, whose universality plays an important role in mathematical research. In this paper, the conditions for the applicability of these theorems are generalised and extended, and the scope of their application is extended from finite closed intervals and open intervals to infinite intervals. Through a series of extension proofs, it is shown that these median theorems still hold in a wider range of intervals. The paper then provides specific applications and explanations of the promoted differential median theorems, demonstrating their convenience and flexibility in solving some specific problems. These generalisations and extensions make the differential median theorems more universal and provide a basis for future higher level mathematical research.*

Keywords: *differential median theorem, Cauchy's median theorem, generalisation of the median theorem*

1. Introduction

The differential median theorem is generally regarded as Rolle's median theorem, Lagrange's median theorem , Cauchy's median theorem, and its conditions of applicability have certain limitations[1]. By extending and expanding the conditions of the existing differential median theorems, these theorems can be made more universal and provide theoretical support for research in a wider range of fields. This series of studies not only helps to deepen the understanding of calculus theory, but also lays a solid foundation for future higher-level mathematical research[5].

2. Proof of promotion

Rolle (Rolle) Median Theorem, Lagrange Median Theorem, Cauchy Median Theorem, all three theorems require that the function $f\left(x\right)$ is continuous on $\left[a,b\right]$ and is derivable in $\left(a,b\right)$ [1][3]. Now we extend the closed interval $\left[a,b\right]$ in the theorem to the infinite interval $\left[a,+\infty\right)$ or $\left(-\infty,+\infty\right)$, and extend the open interval $\left(a,b\right)$ to the infinite interval $\left(a,+\infty\right)$ or $\left(-\infty,+\infty\right)$ can get several corresponding theorems.

Rolle's Extension Theorem 1: If the function $f\left(x\right)$ is continuous on $\left[a,+\infty\right)$, is integrable in $\left(a,+\infty\right)$, and $\lim\limits_{x\to+\infty} f\left(x\right)=f\left(a\right)$, then there exists at least one point $\xi\in\left(a,+\infty\right)$ such that $f'\left(\xi\right)=0$ holds.

Proof:Let the equation about x be $x=\dfrac{1}{n-t}+\dfrac{1}{m-n}+a,t\in\left[m,n\right)$, where m,n is any constant and satisfies $m<n$

Then the parametric equation for t can be obtained

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$$\varphi(t)=\frac{1}{n-t}+\frac{1}{m-n}+a,t\in[m,n) \tag{1}$$

Let $f(x)=f(\varphi(t))=g(t)$, then we have

$$g(m)=f(\varphi(m))=f(a) \tag{2}$$

$$g(n)=\lim_{t\to n^-}g(t)=\lim_{t\to n^-}f(\varphi(t))=\lim_{x\to+\infty}f(x)=f(a) \tag{3}$$

This leads to $g(m)=g(n)$

Since $g(t)$ is continuous on $[m,n]$ and derivable on (m,n) and $g(m)=g(n)$, then by Rolle's theorem, there exists at least one point $\varepsilon\in(m,n)$ such that $g'(\varepsilon)=0$ holds

Then we can make $\xi=\varphi(\varepsilon)$ and thus have

$$g'(\varepsilon)=f'(\xi)\cdot\varphi'(\varepsilon)=0 \tag{4}$$

And because $\varphi'(\varepsilon)=\frac{1}{(n-\varepsilon)^2}\neq 0$, then there must be

$$f'(\xi)=0 \tag{5}$$

In summary, there exists at least one point $\xi\in(a,+\infty)$ such that $f'(\xi)=0$ holds

Rolle's Extension Theorem 2: If $f(x)$ is continuous on $(-\infty,+\infty)$, is integrable in $(-\infty,+\infty)$, and $\lim_{x\to-\infty}f(x)=\lim_{x\to+\infty}f(x)$, then there exists at least one point $\xi\in(-\infty,+\infty)$ such that $f'(\xi)=0$ holds.

Proof:Let the equation about x be $x=\tan\frac{\pi(m+n-2t)}{2(m-n)},t\in(m,n)$, where m,n is any constant and satisfies $m<n$

Then the parametric equation for t can be obtained

$$\varphi(t)=\tan\frac{\pi(m+n-2t)}{2(m-n)},t\in(m,n) \tag{6}$$

Let $f(x)=f(\varphi(t))=g(t)$, then we have

$$g(m)=\lim_{t\to m^+}g(t)=\lim_{t\to m^+}f(\varphi(t))=\lim_{x\to-\infty}f(x) \tag{7}$$

$$g(n)=\lim_{t\to n^-}g(t)=\lim_{t\to n^-}f(\varphi(t))=\lim_{x\to+\infty}f(x) \tag{8}$$

Since $\lim_{x\to-\infty}f(x)=\lim_{x\to+\infty}f(x)$, it follows that $g(m)=g(n)$

Since $g(t)$ is continuous on $[m,n]$ and derivable on (m,n) and $g(m)=g(n)$, then by Rolle's theorem, there exists at least one point $\varepsilon\in(m,n)$ such that $g'(\varepsilon)=0$ holds

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Then we can make $\xi=\varphi(\varepsilon)$ and thus have

$$g'(\varepsilon)=f'(\xi)\cdot\varphi'(\varepsilon)=0 \tag{9}$$

And because $\varphi'(\varepsilon)=-\frac{\pi}{(m-n)\cos^2\frac{\pi(m+n-2\varepsilon)}{2(m-n)}}\neq 0$, then there must be

$$f'(\xi)=0 \tag{10}$$

In summary, there exists at least one point $\xi\in(-\infty,+\infty)$ such that $f'(\xi)=0$ holds

Lagrange's Extension Theorem 1: If $f(x)$ is continuous on $[a,+\infty)$, is integrable in $[a,+\infty)$, and $\lim_{x\to+\infty}f(x)=M$, p are arbitrary nonzero constants, then there exists at least one point $\xi\in(a,+\infty)$ such that $f'(\xi)=\frac{p(f(a)-M)}{[p(a-\xi)+1]^2}$ holds.

Proof: Let the equation $x=\frac{1}{n-t}+\frac{1}{m-n}+a,t\in[m,n)$ about x , where m,n is any constant and satisfies $m<n$

Then the parametric equation for t can be obtained

$$\varphi(t)=\frac{1}{n-t}+\frac{1}{m-n}+a,t\in[m,n) \tag{11}$$

Let $f(x)=f(\varphi(t))=g(t)$, then we have

$$g(m)=f(\varphi(m))=f(a) \tag{12}$$

$$g(n)=\lim_{t\to n^-}g(t)=\lim_{t\to n^-}f(\varphi(t))=\lim_{x\to+\infty}f(x)=M \tag{13}$$

Since $g(t)$ is continuous on $[m,n]$ and derivable on (m,n) , it follows from Lagrange's theorem that there exists at least one point $\varepsilon\in(m,n)$ such that $g'(\varepsilon)=\frac{g(m)-g(n)}{m-n}$ holds, i.e. $g'(\varepsilon)=\frac{f(a)-M}{m-n}$

Then we can make $\xi=\varphi(\varepsilon)=\frac{1}{n-\varepsilon}+\frac{1}{m-n}+a$ and thus have

$$g'(\varepsilon)=f'(\xi)\cdot\varphi'(\varepsilon) \tag{14}$$

And because $\varphi'(\varepsilon)=\frac{1}{(n-\varepsilon)^2}=\left(\frac{(m-n)(a-\xi)+1}{m-n}\right)^2$, then there must be

$$f'(\xi)=\frac{(f(a)-M)(m-n)}{[(m-n)(a-\xi)+1]^2} \tag{15}$$

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Let $p=m-n,(p\neq 0)$, then we get $f'(\xi)=\dfrac{p(f(a)-M)}{[p(a-\xi)+1]^2}$

In summary, there exists at least one point $\xi\in(a,+\infty)$ such that $f'(\xi)=\dfrac{p(f(a)-M)}{[p(a-\xi)+1]^2}$ holds.

Lagrange's Extension Theorem 2: If $f(x)$ is continuous on $[a,+\infty)$, is integrable in $[a,+\infty)$, and $\lim\limits_{x\to+\infty} f(x)=M$, then there exists at least one point $\xi\in(a,+\infty)$ such that $f'(\xi)=\dfrac{2(M-f(a))}{\pi\left[(\xi-a)^2+1\right]}$ holds.

Proof: Let the equation $x=a+\tan\dfrac{\pi(t-m)}{2(n-m)}, t\in[m,n)$ about x , where m,n is any constant and satisfies $m<n$, be set.

Then the parametric equation for t can be obtained

$$\varphi(t)=a+\tan\frac{\pi(t-m)}{2(n-m)}, t\in[m,n) \tag{16}$$

Let $f(x)=f(\varphi(t))=g(t)$, then we have

$$g(m)=f(\varphi(m))=f(a) \tag{17}$$

$$g(n)=\lim_{t\to n^-} g(t)=\lim_{t\to n^-} f(\varphi(t))=\lim_{x\to+\infty} f(x)=M \tag{18}$$

Since $g(t)$ is continuous on $[m,n]$ and derivable on (m,n) , it follows from Lagrange's theorem that there exists at least one point $\varepsilon\in(m,n)$ such that $g'(\varepsilon)=\dfrac{g(m)-g(n)}{m-n}$ holds, i.e. $g'(\varepsilon)=\dfrac{f(a)-M}{m-n}$

Then we can make $\xi=\varphi(\varepsilon)=a+\tan\dfrac{\pi(\varepsilon-m)}{2(n-m)}$ and thus have

$$g'(\varepsilon)=f'(\xi)\cdot\varphi'(\varepsilon) \tag{19}$$

And because $\varphi'(\varepsilon)=\dfrac{\pi}{2(n-m)\cos^2\dfrac{\pi(\varepsilon-m)}{2(n-m)}}=\dfrac{\pi\left[(\xi-a)^2+1\right]}{2(n-m)}$, then there must be

$$f'(\xi)=\frac{2(M-f(a))}{\pi\left[(\xi-a)^2+1\right]} \tag{20}$$

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In summary, there exists at least one point $\xi \in (a,+\infty)$ such that $f'(\xi)=\frac{2(M-f(a))}{\pi\left[(\xi-a)^2+1\right]}$ holds.

Lagrange's Extension Theorem 3: If $f(x)$ is continuous on $(-\infty,+\infty)$ and is integrable in $(-\infty,+\infty)$ and $\lim_{x\to-\infty} f(x)=M$, $\lim_{x\to+\infty} f(x)=N$, there exists m,n an arbitrary constant and satisfying $m<n$, then there exists at least one point $\xi \in (-\infty,+\infty)$ such that $f'(\xi)=\frac{N-M}{\pi(\xi^2+1)}$ holds.

Proof: Let the equation about $x=\tan\frac{\pi(m+n-2t)}{2(m-n)}, t\in(m,n)$

Then the parametric equation for t can be obtained

$$\varphi(t)=\tan\frac{\pi(m+n-2t)}{2(m-n)}, t\in(m,n) \tag{21}$$

Let $f(x)=f(\varphi(t))=g(t)$, then we have

$$g(m)=\lim_{t\to m^+} g(t)=\lim_{t\to m^+} f(\varphi(t))=\lim_{x\to-\infty} f(x)=M \tag{22}$$

$$g(n)=\lim_{t\to n^-} g(t)=\lim_{t\to n^-} f(\varphi(t))=\lim_{x\to+\infty} f(x)=N \tag{23}$$

Since $g(t)$ is continuous on $[m,n]$ and derivable on (m,n) , it follows from Lagrange's theorem that there exists at least one point $\varepsilon \in (m,n)$ such that $g'(\varepsilon)=\frac{g(m)-g(n)}{m-n}$ holds, i.e. $g'(\varepsilon)=\frac{M-N}{m-n}$

Then we can make $\xi=\varphi(\varepsilon)=\tan\frac{\pi(m+n-2\varepsilon)}{2(m-n)}$ and thus have

$$g'(\varepsilon)=f'(\xi)\cdot\varphi'(\varepsilon) \tag{24}$$

And because $\varphi'(\varepsilon)=-\frac{\pi}{(m-n)\cos^2\frac{\pi(m+n-2\varepsilon)}{2(m-n)}}=-\frac{\pi(\xi^2+1)}{m-n}$, then there must be

$$f'(\xi)=\frac{N-M}{\pi(\xi^2+1)} \tag{25}$$

In summary, there exists at least one point $\xi \in (-\infty,+\infty)$ such that $f'(\xi)=\frac{N-M}{\pi(\xi^2+1)}$ holds.

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3. Promotion and application

The generalised differential median theorem provides new methods and ideas for solving a number of problems. The application on infinite intervals can provide a more convenient way to solve specific problems. Through the application of the generalised theorem, the properties of functions can be dealt with more flexibly so as to solve some complex mathematical problems[2][4].

Example 1:Given the functions $f\left(x\right)=\frac{\ln x}{2xe^{x}+x\cos x}$, $x\in\left(0,+\infty\right)$, find $\exists\xi\in\left(0,+\infty\right)$ such that $f'\left(\xi\right)=0$.

Analysis: For this kind of questions we usually use the derivative and find out $f'\left(\xi\right)=0$ when the point ξ , if for the function is not easy to find the derivative, you can also consider using Rolle's promotion theorem. It is easy to find $f\left(1\right)=0$, and it is not difficult to get $\lim\limits_{x\to+\infty}\frac{\ln x}{2xe^{x}+x\cos x}=\lim\limits_{x\to+\infty}\frac{\ln x}{x}\cdot\frac{1}{2e^{x}+\cos x}=0$ by dividing the reference, so we can use Rolle's promotion theorem to solve the problem.

Proof:From the question, we can see that $f\left(x\right)$ is continuous at $\left[0,+\infty\right)$ and is derivable at $\left(0,+\infty\right)$, and it is easy to obtain that

$$\begin{aligned}&f\left(1\right)=0\\&\lim_{x\to+\infty}\frac{\ln x}{2xe^{x}+x\cos x}=\lim_{x\to+\infty}\frac{\ln x}{x}\cdot\frac{1}{2e^{x}+\cos x}=0\end{aligned}\tag{26}$$

resulting in

$$f\left(1\right)=\lim_{x\to+\infty}f\left(x\right)\tag{27}$$

By Rolle's Extension Theorem 1, $\exists\xi\in\left(0,+\infty\right)$, such that $f'\left(\xi\right)=0$, the proof is complete.

Example 2:Let the function $f\left(x\right)$ have the domain of definition $\left(-\infty,+\infty\right)$, which is continuous and derivable in the domain of definition, and satisfy $\frac{1-2x}{1+x^{4}}\le f\left(x\right)\le\frac{3x^{2}-2x+1}{1+x^{4}}$ for $\forall x$,Prove that: $\exists\xi\in R$, such that $f\left(\xi\right)=\frac{\left(1-\xi\right)^{2}}{1+\xi^{4}}$.

ANALYSIS: The problem can be solved by constructing the function $F\left(x\right)=f\left(x\right)-\frac{\left(1-x\right)^{2}}{1+x^{4}}$ transformed into finding $\exists\xi\in R$, such that $F\left(x\right)=0$, which can be solved using Rolle's Theorem Extension.

Proof: The constructor $F\left(x\right)=f\left(x\right)-\frac{\left(1-x\right)^{2}}{1+x^{4}}$, which is continuous and derivable in $\left(-\infty,+\infty\right)$, shows that

$$\frac{-x^{2}}{1+x^{4}}\le F\left(x\right)\le\frac{2x^{2}}{1+x^{4}}\tag{28}$$

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resulting in

$$0=\lim_{x\to-\infty}\frac{-x^2}{1+x^4}\le\lim_{x\to-\infty}F(x)\le\lim_{x\to-\infty}\frac{2x^2}{1+x^4}=0 \tag{29}$$

assume (office)

$$\lim_{x\to-\infty}f(x)=0 \tag{30}$$

again because

$$0=\lim_{x\to+\infty}\frac{-x^2}{1+x^4}\le\lim_{x\to+\infty}f(x)\le\lim_{x\to+\infty}\frac{2x^2}{1+x^4}=0 \tag{31}$$

assume (office)

$$\lim_{x\to+\infty}f(x)=0 \tag{32}$$

the reason why

$$\lim_{x\to-\infty}f(x)=\lim_{x\to+\infty}f(x)=0 \tag{33}$$

By Rolle's Extension Theorem 2, $\exists\xi\in R$, such that $F(\xi)=f(\xi)-\frac{(1-\xi)^2}{1+\xi^4}=0$, i.e. $f(\xi)=\frac{(1-\xi)^2}{1+\xi^4}$, is proved.

Example 3:Let the function $f(x)=\sqrt{x^2+1}-x$, prove that: $\exists\xi\in R$, such that $f'(\xi)=\frac{1}{(\xi-1)^2}$

ANALYSIS: This topic, due to the fact that its proven $f'(\xi)$ structure does not exist π , has a primary term ξ , and $\lim_{x\to-\infty}f(x)\to+\infty$ has no exact value, and considering the existence of $\lim_{x\to+\infty}f(x)$, which has a structure similar to Lagrange's Extension Theorem 1, we make a reasonable attempt by restricting its intervals and adopting the theorem.

Proof: From the function, it is easy to obtain that it is continuous and conductible on the interval . $[0,+\infty)$

on account of

$$f(0)=1 \tag{34}$$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\sqrt{x^2+1}-x=\lim_{x\to+\infty}\frac{1}{\sqrt{x^2+1}+x}=0 \tag{35}$$

By Lagrange's Extension Theorem 1, there exists at least one point $\xi\in(a,+\infty)$ such that $f'(\xi)=\frac{p(f(a)-M)}{[p(a-\xi)+1]^2}=\frac{p}{(p\xi-1)^2}$ holds

Given $p=1$, it follows that there exists at least one point $\xi\in(a,+\infty)$ such that

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$f'(\xi)=\frac{1}{(\xi-1)^2}$ holds.

4. Conclusions

In this paper, the conditions of classical differential median theorems (Rolle's theorem, Lagrange's theorem, Cauchy's theorem)[1] are mainly promoted, which extend the applicability of these theorems on finite closed intervals and open intervals to infinite intervals, and show the value of the application of these promotions in solving the problems of infinite intervals through specific examples, which provide more possibilities for mathematical research[5]. These extension theorems make the differential median theorem more universal, and their applications not only help to solve specific problems, but also can play a role in theoretical research, which is of positive significance to the development of the field of mathematics.

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