

# Research on Control Method of Rotating Inverted Pendulum

Hao Song

School of Electrical and Control Engineering, North University of China, Taiyuan, China

**Abstract:** This paper takes the rotating inverted pendulum system as an example to compare three classical system control methods. In the first half of the article, we use the Lagrangian method to establish the mathematical model of the rotating inverted pendulum system. In the second half of the article, we use the Lyapunov function method, the LQR method, and the PID method to design the control law of the system, so as to realize the self-stable control of the system. By observing the simulation results of the rotating inverted pendulum system under different controllers, we can clearly see the differences among the three control methods and their respective advantages and disadvantages.

**Keywords:** rotating inverted pendulum, Lyapunov function, LQR, PID

## 1. Introduction

The rotating inverted pendulum system, known for its static instability, is a well-known experimental platform to verify the control theory. On the one hand, the structure of the rotary inverted pendulum is simple, which is very suitable for the study of the control scheme, on the other hand, the rotary inverted pendulum has important application in actual life. For example, the attitude control of the aircraft [1] and the balance control of the robot [2] have been widely concerned by researchers at home and abroad.

For the control system of the rotating inverted pendulum, researchers pay attention to many control problems. For example, swing the pendulum from the suspension position to an upright vertical position [3], and stabilize the pendulum in an unstable vertical position [4]. There are a lot of control methods for swing-up and self-stabilization of rotary inverted pendulum. Literature [5-8] uses different control methods to realize the self-stable control of the rotating inverted pendulum near the equilibrium point of the system. In this paper, Lyapunov function method, LQR method and PID method are used to control the system.

## 2. Mathematical Model of Rotating Inverted Pendulum

Ignore various factors such as friction and elastic deformation of mechanical structure, and the rotating arm and vertical pendulum are regarded as light uniform bars. The abstract physical structure of the rotating inverted pendulum system is shown in Figure 1.

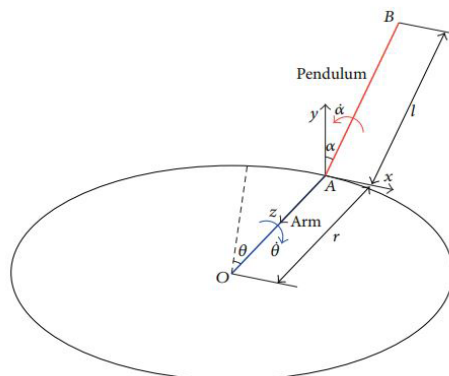


Figure 1: Schematic diagram of Rotating Inverted Pendulum

From Figure 1, we can see that the rotating inverted pendulum is composed of a rotating arm and a pendulum rod mounted on the edge of the arm. The pendulum moves as an inverted pendulum in a plane perpendicular to the rotating arm.

Among them,  $\alpha$  and  $\theta$  are used to describe the inverted pendulum system. The rotation angle of the rotating arm is  $\theta$ , the rotation angle of the driving pendulum is  $\alpha$ , and the corresponding angular velocity is  $\dot{\theta}$  and  $\dot{\alpha}$  respectively. The length of the rotating arm is  $r$ , the geometric point of the pendulum is set to B, and its length to point A is  $l$ .

By applying the Lagrangian method, the nonlinear mathematical model of the rotating inverted pendulum can be obtained:

$$(J_{eq} + mr^2)\ddot{\theta} + mlr \sin(\alpha)\dot{\alpha}^2 - mlr \cos(\alpha)\ddot{\alpha} = T - B_q \dot{\theta} \quad (1)$$

$$\frac{4}{3}ml^2 \ddot{\alpha} - mlr \cos(\alpha)\ddot{\theta} - mgl \sin(\alpha) = 0$$

Where  $V$  is the voltage and  $T$  is the torque, the torque  $T$  is as follows:

$$T = \eta_m \eta_g K_t K_g \frac{V - K_m K_g \dot{\theta}}{R_m} \quad (2)$$

In this paper, the motion of a rotating inverted pendulum near its equilibrium point is studied. Therefore,  $\theta$  and  $\alpha$  in most control time are relatively small. For small values, the approximate formula has  $\cos(\alpha) \approx 1$  and  $\sin(\alpha) \approx \alpha$ . Taking the approximate formula into (1) and solving (1), taking the state  $x = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}^T$  and the voltage  $V$  as the input, the linear model of the rotating inverted pendulum can be obtained:

$$\dot{x} = Ax + BV \quad (3)$$

$$y = Cx + DV$$

Where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bd}{E} & -\frac{cG}{E} & 0 \\ 0 & \frac{ad}{E} & -\frac{bG}{E} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{c\eta_m \eta_g K_t K_g}{R_m E} \\ \frac{b\eta_m \eta_g K_t K_g}{R_m E} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4)$$

With

$$a = J_{eq} + mr^2 \quad (5)$$

$$b = mlr$$

$$c = \frac{4}{3}ml^2$$

$$d = mgl$$

$$E = ac - b^2$$

$$G = \frac{\eta_m \eta_g K_t K_m K_g^2 + B_q R_m}{R_m}$$

Substitute the values in Table 1[8] into (4) to obtain system matrix A and control matrix B.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 41.0435 & -15.3188 & 0 \\ 0 & 83 & -14.667 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 26.8681 \\ 25.7247 \end{bmatrix} \quad (6)$$

*Table 1: Descriptions of physical parameters*

Parameter	Description	Value
$m$	Mass of pendulum arm	0.1675
$r$	Rotating arm length	0.0077
$l$	Length to pendulums center of mass	2.6
$g$	Gravitational constant	0.125
$B_q$	Viscous damping coefficient	9.81
$K_t$	Motor torque constant	70
$K_g$	System gear ratio	0.69
$K_m$	Back-EMF constant	0.215
$R_m$	Armature resistance	0.0040
$\eta_m$	Motor efficiency	0.0077
$\eta_g$	Gear efficiency	0.90
$J_{eq}$	Moment of inertia at the load	0.0033

### 3. Designing Control Law by Lyapunov Functions

#### 3.1 Stabilization of the system

Through calculation, the rank of matrix A is 3. For a linear time invariant system, the rank of the system matrix is less than its order, which means that the equilibrium point of the system is not unique. The full rank controllability matrix  $U_c = [B \ AB \ A^2B \ A^3B]$  shows that the system is completely controllable. Therefore, we can realize the arbitrary configuration of the system poles through the method of state feedback, so as to ensure that the system equilibrium point is unique. The state equation of the system after introducing state feedback can be rewritten as:

$$\dot{x} = (A - BK)x + Br = \tilde{A}x + Br \quad (7)$$

$$y = (C - DK)x + Dr = \tilde{C}x + Dr$$

Where K is the gain of the state feedback,  $r = u + Kx$

In this section, the required closed-loop poles may be set to  $\{-1, -2, -3+4j, -3-4j\}$ . Then we can get the characteristic equation of the closed-loop system.

$$f(s) = s^4 + 9s^3 + 45s^2 + 87s + 50 \quad (8)$$

According to the Cayley-Hamilton theorem, we can get

$$f(A) = \tilde{A}^4 + 9\tilde{A}^3 + 45\tilde{A}^2 + 87\tilde{A} + 50 \quad (9)$$

By formula(10), the gain vector K is  $[-0.0426 \quad 5.0203 \quad -0.6442 \quad 0.4273]$ .

$$K = [0 \quad 0 \quad 0 \quad 1]U_c^{-1}f(A) \quad (10)$$

### 3.2 Designing of system control law

According to the Lyapunov stability theorem, the design of control laws using the Lyapunov method can follow the process:

(1) Construct a function that satisfies the conditions of the Lyapunov function  $V(x,t)$ , whose the first-order continuous derivable with respect to x is not less than zero and only when . The Lyapunov function selected in this paper is a quadratic function:  $V(x,t) = x^T P x$ .

The solution of the energy matrix P is as follows.

$$\tilde{A}^T P + P \tilde{A} = -Q \quad (11)$$

Where Q is the identity matrix.

(2) Find the first derivative with respect to time  $\dot{V}(x,t)$ .

$$\dot{V}(x,t) = (\dot{x}^T P x + x^T P \dot{x}) = x^T (\tilde{A}^T P + P \tilde{A})x + (B^T P x + x^T P B)r \quad (12)$$

$$\dot{V}(x,t) = x^T (\tilde{A}^T P + P \tilde{A})x + 2x^T P B r$$

(3) The control law is designed so that the inequality  $\dot{V}(x,t) \leq 0$  is valid only at point  $x_e$ .

Let  $\dot{V}(x,t) \leq 0$ , the control law r is obtained as

$$r = -\frac{x^T (\tilde{A}^T P + P \tilde{A})x}{2x^T P B} - k 2x^T P B \quad (13)$$

Where k is any positive number, substitute (12) into (11) to get.

$$\dot{V}(x,t) = -k (2x^T P B)^2 < 0, \quad x \neq 0 \quad (14)$$

Let  $k = 0.02$ , the initial state of the system is  $x_0 = [0 \quad 1 \quad 0 \quad 0]^T$ , using Matlab to simulate the rotating inverted pendulum system with the control law introduced. The results of the system control law are shown in Figure 2, and the results of the system state are shown in Figure 3.

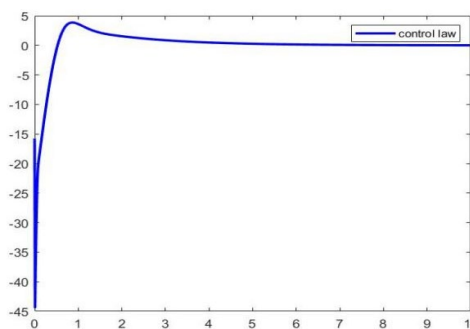


Figure 2: Control law curve

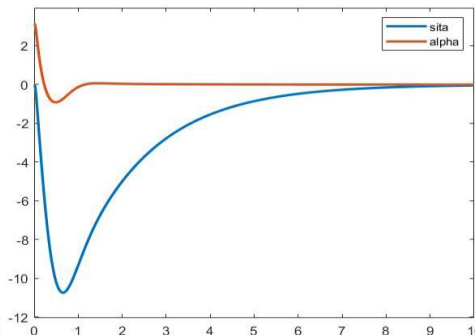


Figure 3: State variable curve

The results in figures 2 and 3 show that under the control laws designed by the Lyapunov function, the control law and the state variable of the system converge to zero to realize the task of stable control of the inverted pendulum system.

#### 4. Designing Control Law by linear quadratic regulator (LQR)

The basic idea of using the LQR method to design the control law is as follows. For a control system, we first select a system cost function, and then find a control law (On the one hand, the control system is stable, on the other hand, the system performance function takes an extreme value).

For the linear steady-state system (3) in this paper, we could set the system performance function as.

$$J = \int_0^{\infty} (x^T Qx + u^T Ru + 2x^T Nu) dt \quad (15)$$

Then, the feedback control law that minimizes J is as following.

$$u = -Kx \quad (16)$$

With

$$K = R^{-1}(B^T P + N^T) \quad (17)$$

Where P can be obtained by the continuous-time algebraic Riccati equation (18).

$$A^T P + PA - (PB + N)R^{-1}(B^T P + N^T) + Q = 0 \quad (18)$$

It is worth mentioning that when Q is a diagonal matrix.

$$x^T Qx = q_1 x_1^2 + q_2 x_2^2 + \dots + q_n x_n^2 \quad (19)$$

Therefore, when Q is a diagonal matrix, we can change the influence of the corresponding state variables on the system performance function J by adjusting the values of the elements in Q. In other words, during the convergence process of the system performance function J, the state variable corresponding to the larger element in Q should be given priority.

The same is true for the matrices R and N in the system performance function J.

When designing an LQR controller, the matrices Q, R, and N should be chosen independently. The Q matrix often takes a diagonal matrix. For the inverted pendulum system of text, when R is set to 1,  $N = [1 \ 1 \ 0 \ 0]^T$ , through the method of parameter tuning, we obtain that the control effect of the system is best when the elements on the diagonal of the Q matrix are nearby  $[2400 \ 30 \ 1 \ 1]$ . The rotating inverted pendulum system with the introduction of LQR controller is simulated by Matlab. The system control law results are shown in Figure 4, and the system state results are shown in Figure 5.

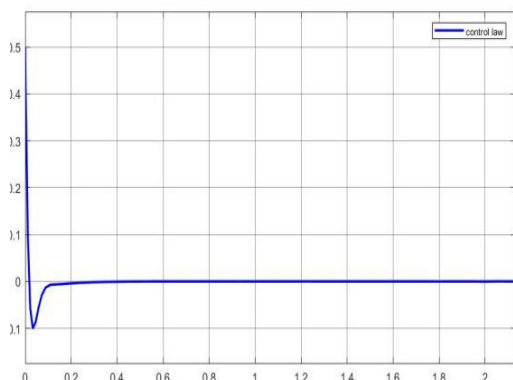


Figure 4: Control law curve

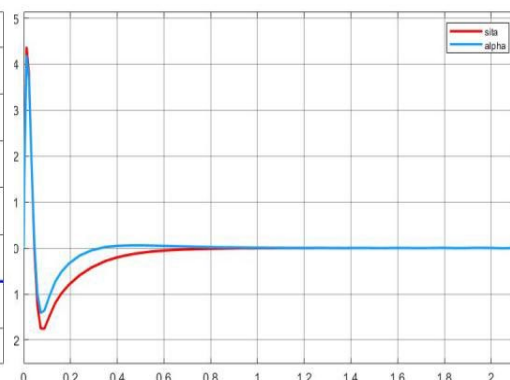


Figure 5: State variable curve

The results in Figures 4 and 5 show that after introducing the controller using the LQR method, the

system control law and state variables eventually converge to zero, and the task of stable control of the inverted pendulum system is also achieved.

## 5. Designing Control Law by Proportional Integral Derivative (PID)

This section introduces a PID controller to the system based on Model (7). The parameter tuning of PID controller is the core of control system design. We use engineering tuning method to adjust the proportional coefficient, integral coefficient and differential coefficient of PID controller.

### 5.1 Determine Proportional Gain $K_p$

When determining the proportional gain  $K_p$ , we remove the integral term and differential term of the PID, and then gradually increase the proportional gain from zero. During the experiment, we found that, during  $K_p < 1 \cap K_p > 1.1$ , the system oscillation intensified. The experimental result is shown in Fig. 6.

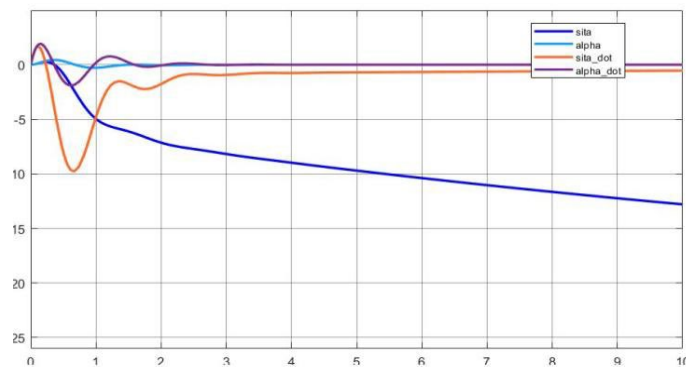


Figure 6: State variable curve of the system

When the proportional gain  $K_p$  is around one, the system oscillation is the most moderate. The experimental results are shown in Figure 7.

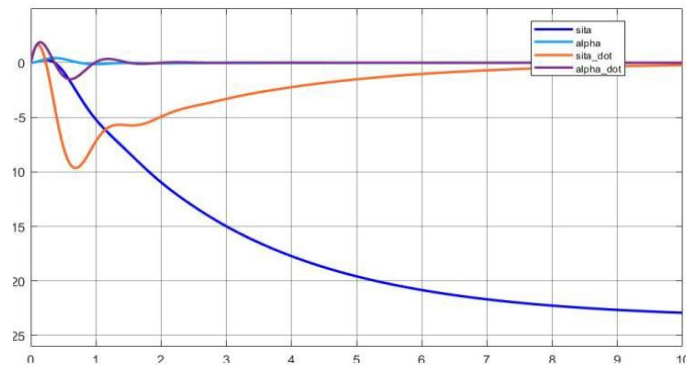


Figure 7: State variable curve of the system

### 5.2 Determine Integral Gain $K_i$

When the proportional gain  $K_p$  has been determined, we gradually increase the proportional gain  $K_i$  from zero until the system will oscillate. Then, we gradually reduce the proportional gain  $K_i$  until the system oscillation disappears. During the experiment, we found that the control effect is best when the proportional gain is near 0.3. The experimental results are shown in Figure 8.

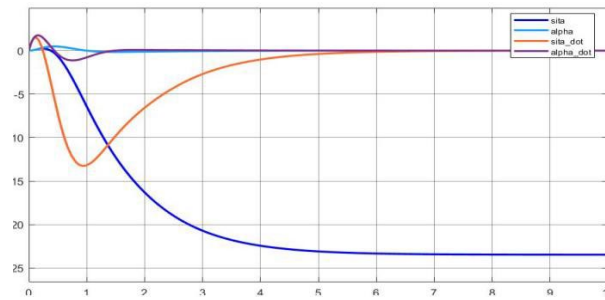


Figure 8: State variable curve of the system

### 5.3 Determine Differential Gain $K_D$

The differential gain  $K_D$  generally does not need to be set, and the value of the differential gain can be set to 0.

## 6. Conclusion

Through the simulation results, we can see that the three control methods can complete the control task of the inverted pendulum system well. The advantages and disadvantages of the three control methods are briefly summarized below.

The method of using Lyapunov function to design control law is suitable for both linear systems and nonlinear systems, and has a very wide range of applications. Moreover, the design method could not only reflect the specific control relationship within the system, but also achieve a satisfactory control effect. However, when using Lyapunov functions to design control laws, it is often necessary to design different Lyapunov functions for different control systems, which undoubtedly increases the design task and difficulty.

The controller design method using linear quadratic regulator (LQR) is unique in both SISO and MIMO systems. In other words, for any set of required matrices Q, R and N, we can always find a unique feedback matrix for the system. However, in the process of design, how to reasonably value the matrix is obviously a difficult problem.

Using the proportional integral derivative (PID) method to design the controller, in the controller parameter setting part, the operation is relatively simple. However, this design method cannot reflect the specific control relationship within the system, and due to the limitation of its own algorithm, its control effect is difficult to achieve the desired effect.

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