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Cramér-Type Moderate Deviations for Self-Attracting Diffusion Processes Driven by Weighted Fractional Brownian Motion

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Abstract: *This paper studies the deviation properties of certain quadratic functionals in a linear self-attracting diffusion process driven by weighted fractional Brownian motion, including deviation inequalities and Cramér-type moderate deviations. On this basis, we further derive the Cramér-type moderate deviations for the least squares estimator of the drift parameter and its self-normalized form. The main methods employed in this paper are deviation inequalities for multiple Wiener–Itô integrals and Cramér-type moderate deviations. The analysis covers both the subcritical and critical cases within the parameter range considered. The proofs rely on Wiener–Itô decompositions, deterministic asymptotics, kernel norm estimates, and contraction bounds to identify the dominant random terms and control the remaining contributions. The resulting Gaussian tail approximations complement existing strong consistency and asymptotic distribution results, providing a more precise description of the tail behavior of the quadratic functionals and the estimator beyond the central limit regime.*

Keywords: *Weighted Fractional Brownian Motion, Self-attracting Diffusion, Cramér-type Moderate Deviations, Wiener–Itô Chaos*

1. Introduction

Motivated by recent work on stochastic models driven by fractional-type Gaussian noises, we consider a linear self-attracting diffusion driven by weighted fractional Brownian motion:

$$Y_t^{a,b} := \int_0^t \left(X_t^{a,b} - X_s^{a,b}\right) ds, \qquad t \geq 0,$$

and

$$dX_t^{a,b} = dB_t^{a,b} - \theta Y_t^{a,b}\, dt + \nu\, dt, \qquad t \geq 0, \tag{1}$$

where $X_0^{a,b} = 0$, $\theta > 0$, $\nu \in \mathbb{R}$, and the parameters a and b satisfy

$$a > -1, \quad |b| < 1 \wedge (1 + a).$$

This model extends the classical linear self-attracting diffusion to a weighted-fractional-noise setting. The memory term $\int_0^t \left(X_t^{a,b} - X_s^{a,b}\right) ds$ describes attraction toward the past trajectory, while the weighted fractional Brownian motion brings in a nonstationary covariance structure. Together, these features make the asymptotic analysis of the associated statistical functionals more delicate.

1.1. Background

The driving noise $B^{a,b} = \{B_t^{a,b}, t \geq 0\}$, introduced by Bojdecki, Gorostiza and Talarczyk[5], is a centered Gaussian process with covariance

$$\mathbb{E}\left[B_t^{a,b} B_s^{a,b}\right] = K_{a,b} \int_0^{s\wedge t} u^a \left((t-u)^b + (s-u)^b\right) du, s, t \geq 0.$$

Path properties, quadratic variations and statistical inference problems for weighted fractional Brownian motion and related models have been examined in earlier studies[4, 8–10]. Under continuous

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observation of $\{X_t^{a,b}, 0 \le t \le T\}$, least squares provides a natural way to estimate the unknown parameters. Hu, Nualart and Zhou have also applied least squares methods to fractional Ornstein–Uhlenbeck processes[6]. To express the drift least squares estimator, define the following functionals.

$$\Phi_T(a,b) = T\int_0^T \left(Y_t^{a,b}\right)^2 \, dt - \left(\int_0^T Y_t^{a,b} \, dt\right)^2$$

and

$$N_T(a,b) = B_T^{a,b}\int_0^T Y_t^{a,b} \, dt - T\int_0^T Y_t^{a,b} \, \delta B_t^{a,b}.$$

Here $\int_0^T Y_t^{a,b} \, \delta B_t^{a,b}$ denotes the Skorohod integral with respect to $B^{a,b}$. The relevant Gaussian stochastic calculus and Malliavin calculus frameworks are presented by Alós, Mazet and Nualart and by Nualart [3, 7].

The drift least squares estimator then satisfies

$$\hat{\theta}_T - \theta = \frac{N_T(a,b)}{\Phi_T(a,b)}. \tag{2}$$

This paper continues the work of Yan, Guo and Pei[2], where the strong consistency and asymptotic distributions of $\hat{\theta}_T$ and $\hat{v}_T$ were established for the same model. Their results do not quantify tail probabilities on moderate deviation scales. For $0 < b \le \frac{1}{2}$ and $a + b > 0$, we establish a deviation inequality for $\Phi_T(a,b)$, Cramér-type moderate deviations for $N_T(a,b)$ and $\Phi_T(a,b)$, and standard and self-normalized Cramér-type moderate deviations for $\hat{\theta}_T$.

1.2. Main results

This section states the Cramér-type moderate deviations for $N_T(a,b)$ and $\Phi_T(a,b)$, together with the standard and self-normalized Cramér-type moderate deviations for $\hat{\theta}_T$.

Theorem 1.1. *Assume* $0 < b \le \frac{1}{2}$ *and* $a + b > 0$. *Then there exist constants* $C_1, C_2 > 0$, *independent of* T *and* x, *such that the following estimates hold for all* $x > 0$ *and all sufficiently large* T.

If $0 < b < \frac{1}{2}$, *then*

$$\mathbb{P}\left(T^{-3-a+b}\left|\Phi_T(a,b) - \frac{\Gamma(2+a+b)}{2(a+2-b)\Gamma(a)\theta^{1+b}}T^{3+a-b}\right| \ge x\right) \le C_1\exp\{-C_2 T x min\{Tx, \eta_T\}\}. \tag{3}$$

If $b = \frac{1}{2}$, *then*

$$\mathbb{P}\left(\left|T^{-a-\frac{5}{2}}\Phi_T\left(a,\frac{1}{2}\right) - \frac{\Gamma\left(a+\frac{3}{2}\right)}{2\Gamma(a)\theta^{3/2}}\right| \ge x\right) \le C_1\exp\left\{-C_2\frac{Tx}{\sqrt{logT}}\min\left\{\frac{Tx}{\sqrt{logT}}, \eta_T\right\}\right\}. \tag{4}$$

Here

$$\eta_T = \begin{cases} T^{1/2}, & 0 < b < \frac{1}{4}, \\ T^{1/2}(logT)^{-1/4}, & b = \frac{1}{4}, \\ T^{1-2b}, & \frac{1}{4} < b < \frac{1}{2}, \\ \sqrt{logT}, & b = \frac{1}{2}. \end{cases}$$

Theorem 1.1 is a deviation inequality for $\Phi_T(a,b)$. It controls the fluctuation of this functional around its deterministic leading term through an explicit exponential bound.

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Theorem 1.2. *Assume* $0 < b \le \frac{1}{2}$ *and* $a + b > 0$. *Let* λ_T *be positive with* $\lambda_T \to \infty, \frac{\lambda_T^3}{\eta_T} \to 0$. *For each* $\rho > 0$, *uniformly for* $0 < x \le \rho\lambda_T$, *as* $T \to \infty$, *we have:*

If $0 < b < \frac{1}{2}$, *then*

$$\sup_{0\le x\le \rho\lambda_T}\left|\frac{\mathbb{P}\left(\pm\frac{\sqrt{2}\,\theta^{b+\frac{1}{2}}}{\alpha_{a,b}\sqrt{\beta_{a,b}}\,T^{2+a-b}}\left[N_T(a,b)-\frac{2\alpha_{a,b}B(a+1,b)}{\theta(a+b+1)}T^{1+a+b}\right]\ge x\right)}{1-\Phi(x)}-1\right|\to 0,$$

and

$$\sup_{0\le x\le \rho\lambda_T}\left|\frac{\mathbb{P}\left(\pm\frac{\sqrt{2}\,\theta^{b+\frac{3}{2}}}{\alpha_{a,b}\sqrt{\beta_{a,b}}\,T^{2+a-b}}\left[\Phi_T(a,b)-\frac{\Gamma(2+a+b)}{2(a+2-b)\Gamma(a)\theta^{1+b}}T^{3+a-b}\right]\ge x\right)}{1-\Phi(x)}-1\right| \to 0. \tag{5}$$

If $b = \frac{1}{2}$, *then*

$$\sup_{0\le x\le \rho\lambda_T}\left|\frac{\mathbb{P}\left(\pm\frac{\theta\sqrt{1+2a}}{2\sqrt{2}\,\alpha_{a,\frac{1}{2}}T^{a+\frac{3}{2}}\sqrt{logT}}\left[N_T\left(a,\frac{1}{2}\right)-\frac{2\alpha_{a,\frac{1}{2}}B\left(a+1,\frac{1}{2}\right)}{\theta\left(a+\frac{3}{2}\right)}T^{a+\frac{3}{2}}\right]\ge x\right)}{1-\Phi(x)}-1\right|\to 0,$$

and

$$\sup_{0\le x\le \rho\lambda_T}\left|\frac{\mathbb{P}\left(\pm\frac{\theta^2\sqrt{1+2a}}{2\sqrt{2}\,\alpha_{a,\frac{1}{2}}T^{a+\frac{3}{2}}\sqrt{logT}}\left[\Phi_T\left(a,\frac{1}{2}\right)-\frac{\Gamma\left(a+\frac{5}{2}\right)}{2\left(a+\frac{3}{2}\right)\Gamma(a)\theta^{3/2}}T^{a+\frac{5}{2}}\right]\ge x\right)}{1-\Phi(x)}-1\right|\to 0. \tag{6}$$

Theorem 1.2 gives Gaussian-type tail approximations for $N_T(a,b)$ and $\Phi_T(a,b)$ on moderate deviation scales. It gives a sharper description of rare fluctuations than the central limit theorem and provides the probability estimates needed for the estimator $\hat{\theta}_T$.

As an application, we obtain the standard and self-normalized Cramér-type moderate deviations for the estimator $\hat{\theta}_T$.

Theorem 1.3. *Under the assumptions of Theorem 1.2, for each* $\rho > 0$, *uniformly for* $0 < x \le \rho\lambda_T$, *as* $T \to \infty$, *the following estimates hold.*

If $0 < b < \frac{1}{2}$, *then*

$$\left|\frac{\mathbb{P}\left(\pm\frac{\Gamma(2+a+b)}{\sqrt{2}\,(a+2-b)\Gamma(a)\alpha_{a,b}\sqrt{\beta_{a,b}}\sqrt{\theta}}T\left(\hat{\theta}_T-\theta\right)\ge x\right)}{1-\Phi(x)}-1\right|\to 0,$$

and

$$\left|\frac{\mathbb{P}\left(\pm\frac{\sqrt{2}\,\theta^{b+\frac{1}{2}}}{\alpha_{a,b}\sqrt{\beta_{a,b}}\,T^{2+a-b}}\Phi_T(a,b)\left(\hat{\theta}_T-\theta\right)\ge x\right)}{1-\Phi(x)}-1\right|\to 0. \tag{7}$$

If $b = \frac{1}{2}$, *then*

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$$\left|\frac{\mathbb{P}\left(\pm\frac{\Gamma\left(a+\frac{5}{2}\right)\sqrt{1+2a}}{4\sqrt{2}\,\left(a+\frac{3}{2}\right)\Gamma(a)\alpha_{a,\frac{1}{2}}\sqrt{\theta}}\frac{T}{\sqrt{logT}}\left(\hat{\theta}_T-\theta\right)\geq x\right)}{1-\Phi(x)}-1\right|\to 0,$$

and

$$\left|\frac{\mathbb{P}\left(\pm\frac{\theta\sqrt{1+2a}}{2\sqrt{2}\,\alpha_{a,\frac{1}{2}}T^{a+\frac{3}{2}}\sqrt{logT}}\Phi_T\left(a,\frac{1}{2}\right)\left(\hat{\theta}_T-\theta\right)\geq x\right)}{1-\Phi(x)}-1\right|\to 0. \tag{8}$$

2. Wiener–Itô Decomposition and Important Parameter Estimates

This section collects the estimates needed in the proofs of the main results. We decompose the basic functionals into deterministic terms and Wiener chaoses, and identify the deterministic scales, the dominant second moments and the contraction bounds required for the Cramér-type estimates.

Throughout this section, $0<b\leq 1/2$ and $a+b>0$. Let $\mathcal{H}=\mathcal{H}_{a,b}$ be the Hilbert space associated with $B^{a,b}$. For step functions, its inner product is given by

$$\langle f,g\rangle_{\mathcal{H}}=\alpha_{a,b}\int_0^T\int_0^T f(t)g(s)(t\wedge s)^a|t-s|^{b-1}\,ds\,dt,$$

where

$$\alpha_{a,b}=\frac{b}{2\mathbb{B}(a+1,b+1)}.$$

For symmetric kernels $f\in\mathcal{H}^{\odot p}$, the multiple Wiener–Itô integrals with respect to $B^{a,b}$ satisfy

$$\mathbb{E}\left[I_p^{a,b}(f)I_q^{a,b}(g)\right]=\begin{cases}p!\,\langle f,g\rangle_{\mathcal{H}^{\otimes p}}, & p=q,\\ 0, & p\neq q.\end{cases}$$

By the variation of the constant method applied to (1),

$$Y_t^{a,b}=e^{-\frac{\theta t^2}{2}}\int_0^t s\,e^{\frac{\theta s^2}{2}}\delta B_s^{a,b}+\frac{\nu}{\theta}\left(1-e^{-\frac{\theta t^2}{2}}\right),\qquad t\geq 0.$$

We begin with the Wiener–Itô decomposition of $\Phi_T(a,b)$.

Proposition 1. *Let*

$$h_t(s)=se^{\frac{\theta}{2}(s^2-t^2)}\mathbf{1}_{[0,t]}(s),\qquad \xi_T(s)=se^{\frac{\theta}{2}s^2}G_T(s),$$

where

$$G_T(s)=\int_s^T e^{-\frac{\theta r^2}{2}}\,dr,\qquad H_T(s)=\int_s^T e^{-\theta r^2}\,dr,\quad c_T=\int_0^T\Lambda_\theta(t)\,dt.$$

then

$$\Phi_T(a,b)=I_1\left(\psi_{2,T}^{a,b}\right)+I_2\left(\varphi_{2,T}^{a,b}\right)+D_T(a,b),$$

where

$$\psi_{2,T}^{a,b}(s)=\frac{2\nu}{\theta}\left(Tse^{\frac{\theta s^2}{2}}[G_T(s)-H_T(s)]-c_T\xi_T(s)\right),$$

$$\varphi_{2,T}^{a,b}(s,t)=st\,e^{\frac{\theta}{2}(s^2+t^2)}\left(TH_T(s\vee t)-G_T(s)G_T(t)\right),$$

and

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$$D_T(a,b) = \alpha_{a,b}\int_{[0,T]^2} s\,t\,e^{\frac{\theta}{2}(s^2+t^2)}[TH_T(s\vee t) - G_T(s)G_T(t)](s\wedge t)^a|s-t|^{b-1}\,ds\,dt + \frac{\nu^2}{\theta^2}(TH_T(0) - [G_T(0)]^2). \tag{9}$$

Proof. Applying the product formula for Wiener integrals and collecting the deterministic, first-chaos and second-chaos components gives the denominator decomposition (9).

The numerator has the corresponding Wiener–Itô decomposition

$$N_T(a,b) = d_T(a,b) + I_1\big(\psi_{1,T}^{a,b}\big) + I_2\big(\varphi_{1,T}^{a,b}\big),$$

where

$$\psi_{1,T}^{a,b}(t) = \frac{\nu}{\theta}\big(c_T - T\Lambda_\theta(t)\big),$$

$$\varphi_{1,T}^{a,b}(s,t) = \frac{1}{2}\big(\xi_T(s) + \xi_T(t)\big) - \frac{T}{2}(s\wedge t)e^{-\frac{\theta}{2}|t^2-s^2|},$$

and

$$d_T(a,b) = \alpha_{a,b}\int_{[0,T]^2} \xi_T(s)(t\wedge s)^a|t-s|^{b-1}\,ds\,dt.$$

We now estimate the deterministic terms $d_T(a,b)$ and $D_T(a,b)$. These terms are the non-random parts in the decompositions of $N_T(a,b)$ and $\Phi_T(a,b)$, and their leading orders determine the centering scales used in the subsequent normalizations.

Proposition 2. *Assume that* $0 < b \le \frac{1}{2}$ *and* $a + b > 0$. *Then, as* $T \to \infty$,

$$d_T(a,b) \sim \frac{2\alpha_{a,b}B(a+1,b)}{\theta(a+b+1)}T^{1+a+b}. \tag{10}$$

Moreover,

$$D_T(a,b) \sim \frac{\Gamma(2+a+b)}{2(2-b+a)\Gamma(a)\theta^{1+b}}T^{3+a-b}.$$

Proof. Let $m_{1,T}(a,b)$ denote the right-hand side of (10).

For $D_T(a,b)$, write $D_T = D_{1,T} - D_{2,T} + D_{3,T}$, where the principal contribution comes from $D_{1,T}$. After the change of variables u=t-s, the standard Laplace estimate gives

$$D_{1,T} \sim \frac{\alpha_{a,b}\Gamma(b)}{\theta^{1+b}(a+2-b)}T^{a+3-b} = \frac{\Gamma(2+a+b)}{2(a+2-b)\Gamma(a)\theta^{1+b}}T^{a+3-b}.$$

Direct estimates give

$$D_{2,T} = O(T^{a+b+1}), \qquad D_{3,T} = O(T).$$

Since $0 < b \le 1/2$ and $a + b > 0$, both terms are of lower order than T^{a+3-b}.

The main random contributions are carried by $\varphi_{1,T}^{a,b}$ and $\varphi_{2,T}^{a,b}$. We first determine their precise norm asymptotics, and then give a rough bound for $\psi_{1,T}^{a,b}$ and $\psi_{2,T}^{a,b}$, whose norms are of lower order and will not affect the principal normalization.

Proposition 3. *Assume* $0 < b \le \frac{1}{2}$. *Then, as* $T \to \infty$,

$$\| \varphi_{1,T}^{a,b} \|_{\mathcal{H}}^2 = \begin{cases} \frac{\alpha_{a,b}^2\beta_{a,b}}{4\theta^{2b+1}}T^{4+2a-2b} + O(T^{3+2a}), & 0 < b < \frac{1}{2}, \\ \frac{4\alpha_{a,\frac{1}{2}}^2}{(1+2a)\theta^2}T^{3+2a}logT + O\big(T^{3+2a}\sqrt{logT}\big), & b = \frac{1}{2}, \end{cases} \tag{11}$$

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and

$$\| \varphi_{2,T}^{a,b} \|_{\mathcal{H}}^2 = \begin{cases} \dfrac{\alpha_{a,b}^2 \beta_{a,b}}{4\theta^{3+2b}} T^{4+2a-2b} + O(T^{3+2a}), & 0 < b < \dfrac{1}{2}, \\ \dfrac{4\alpha_{a,\frac{1}{2}}^2}{(1+2a)\theta^4} T^{3+2a} logT + O\left(T^{3+2a}\sqrt{logT}\right), & b = \dfrac{1}{2}. \end{cases} \tag{12}$$

where

$$\beta_{a,b} = \frac{2b+1}{1+a-b}\Gamma(b)\left(\Gamma(b) + \frac{2\Gamma(1-2b)\Gamma(2b)}{\Gamma(1-b)}\right).$$

Proof. Set

$$\begin{aligned} G_{a,b}(T) \quad = & \int_{[0,T]^4} (s \wedge t)\,(u \wedge v) e^{-\frac{\theta}{2}|s^2-t^2|} e^{-\frac{\theta}{2}|u^2-v^2|} \\ & \times (s \wedge u)^a |s-u|^{b-1} (t \wedge v)^a |t-v|^{b-1}\, ds\, dt\, du\, dv. \end{aligned} \tag{13}$$

The exponential factors in (13) localize the main contribution near the diagonal and the terminal boundary. The changes of variables used in Lemmas 4.3 and 4.6 of Yan, Guo and Pei[2] reduce the singular part to a one-dimensional convolution, with a logarithmic correction when b=1/2. Substitution into the localized representation of $G_{a,b}(T)$, together with the constants computed by Yan, Guo and Pei[2], yields

$$G_{a,b}(T) = \begin{cases} \beta_{a,b}\theta^{-1-2b} T^{2+2a-2b} + O(T^{1+2a}), & 0 < b < \dfrac{1}{2}, \\ \dfrac{16}{\theta^2(1+2a)} T^{1+2a} \log T + O(T^{1+2a}), & b = \dfrac{1}{2}. \end{cases} \tag{14}$$

We first prove (12). Write

$$\varphi_{2,T}^{a,b}(s,t) = A_T(s,t) - B_T(s,t), \tag{15}$$

where

$$A_T(s,t) = Tst\, e^{\frac{\theta}{2}(s^2+t^2)} H_T(s \vee t), \qquad B_T(s,t) = st\, e^{\frac{\theta}{2}(s^2+t^2)} G_T(s) G_T(t).$$

Using the finite-endpoint expansion of H_T, we decompose A_T as

$$A_T(s,t) = \tilde{A}_T(s,t) + R_T(s,t), \qquad \tilde{A}_T(s,t) = \frac{T}{2\theta}(s \wedge t) e^{-\frac{\theta}{2}|s^2-t^2|}, \tag{16}$$

with

$$|R_T(s,t)| \le CT \frac{s \wedge t}{(s \vee t)^2} e^{-\frac{\theta}{2}|s^2-t^2|} + Cst\, e^{-\theta T^2 + \frac{\theta}{2}(s^2+t^2)}. \tag{17}$$

By (13)-(14), the leading term has the asymptotic order and constant appearing in (12). The same localization argument shows that the remainder and its cross term with the leading kernel are of lower order. The remaining kernel is uniformly bounded, so its norm and cross terms are also of lower order by the beta-kernel estimate and Cauchy's inequality. The estimates for $\tilde{A}_T$, R_T, and B_T then prove (12). The estimate (11) is obtained in the same way from the corresponding decomposition of $\varphi_{1,T}^{a,b}$.

Lemma 4. *Assume that* $0 < b \le \frac{1}{2}$. *Then, as* $T \to \infty$, *for* $i = 1,2$,

$$\| \psi_{i,T}^{a,b} \|_{\mathcal{H}}^2 = O\left(T^{\max\{2,a+b+1\}}\right).$$

Proof. Proposition 1 implies $\left|\psi_{i,T}^{a,b}(s)\right| \le C\left(1 + Te^{-\frac{\theta s^2}{2}}\right), \quad i = 1,2.$ Integrating this bound against the covariance kernel yields the stated estimate.

It remains to estimate the first contraction norms of $\varphi_{1,T}^{a,b}$ and $\varphi_{2,T}^{a,b}$. These bounds control the non-Gaussian part of the double Wiener integrals generated by these kernels.

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Proposition 5. *Assume* $0 < b \le \frac{1}{2}$. *For* $i = 1,2$, *the kernels* $\varphi_{i,T}^{a,b}$ *satisfy*

$$\| \varphi_{i,T}^{a,b} \otimes_1 \varphi_{i,T}^{a,b} \|_{\mathcal{H}^{\otimes 2}}^2 = \begin{cases} O(T^{6+4a-4b}), & 0 < b < \frac{1}{4}, \\ O(T^{5+4a} logT), & b = \frac{1}{4}, \\ O(T^{4+4a+4b}), & \frac{1}{4} < b \le \frac{1}{2}. \end{cases}$$

Proof. Since $\tilde{A}_T(s,t) = \frac{T}{2\theta} K_T(s,t)$, it suffices to estimate $\| K_T \otimes_1 K_T \|_{\mathcal{H}^{\otimes 2}}$. Since the value of $\theta > 0$ only affects constants, we take $\theta = 2$ in the following power estimate. The contraction kernel has the following representation:

$$Q_T(s,t) = \int_0^T \int_0^T (s \wedge u)\,(t \wedge v) e^{-|s^2-u^2|} e^{-|t^2-v^2|} (u \wedge v)^a |u-v|^{b-1}\, du\, dv.$$

For $s, t \in [R, 2R]$ and $R \ge 1$, the exponential kernels localize the integration variables within order R^{-1} of s and t. Rescaling these neighborhoods gives

$$|Q_T(s,t)| \le C R^a (|s-t| + R^{-1})^{b-1}, \qquad s,t \in [R,2R],\ R \ge 1. \tag{18}$$

Let I_R denote the same-shell contribution to the squared contraction norm from $[\mathrm{R}, 2\mathrm{R}]^4$. Applying (18) to the contraction kernels and the covariance factors gives

$$I_R \le C \begin{cases} R^{4a+2-4b}, & 0 < b < \frac{1}{4}, \\ R^{4a+1} \log R, & b = \frac{1}{4}, \\ R^{4a+4b}, & \frac{1}{4} < b \le \frac{1}{2}. \end{cases} \tag{19}$$

The same comparison controls mixed shells. Summing (19) over dyadic $1 \le R \le T$ under $a + b > 0$ and using $\tilde{A}_T(s,t) = \frac{T}{2\theta} K_T(s,t)$ gives the three principal-kernel orders stated in Proposition 5.

Combining (16), the bound $|B_T(s,t)| \le C$, the standard beta-kernel integral estimate, and Cauchy's inequality for contractions gives

$$\| (R_T - B_T) \otimes_1 (R_T - B_T) \|_{\mathcal{H}^{\otimes 2}}^2 = O(T^{4+4a+4b}). \tag{20}$$

Finally, Cauchy's inequality combines this principal-kernel bound with (20) to give Proposition 5.

The estimate for $\varphi_{1,T}^{a,b}$ follows in the same way. This completes the proof.

3. Proofs of Main Results

This section proves the main results. We first use the decompositions and kernel estimates for the two functionals to establish the deviation inequality for $\Phi_T(a,b)$ and the Cramér-type moderate deviations for $N_T(a,b)$ and $\Phi_T(a,b)$. We then apply these functional-level deviation results to the least squares estimator and obtain the standard and self-normalized Cramér-type moderate deviations for $\hat{\theta}_T$.

3.1. Proof of Theorem 1.1

By the norm estimates (11) and (12), the standard-deviation scale of the second-order terms is given by

$$S_T = \begin{cases} T^{2+a-b}, & 0 < b < \frac{1}{2}, \\ T^{a+\frac{3}{2}} \sqrt{\log T}, & b = \frac{1}{2}, \end{cases} . \tag{21}$$

For $i = 1,2$, define $F_{i,T}(a,b) = I_2\big(\varphi_{i,T}^{a,b}\big) / \big(\sigma_{i,a,b} S_T\big)$, where the constants $\sigma_{i,a,b}$ are determined by the leading terms in (11) and (12), namely

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$$\sigma_{1,a,b}^2 = \begin{cases} \frac{\alpha_{a,b}^2 \beta_{a,b}}{2\theta^{2b+1}}, & 0 < b < \frac{1}{2}, \\ \frac{8\alpha_{a,\frac{1}{2}}^2}{(1+2a)\theta^2}, & b = \frac{1}{2}, \end{cases} \quad \sigma_{2,a,b}^2 = \begin{cases} \frac{\alpha_{a,b}^2 \beta_{a,b}}{2\theta^{3+2b}}, & 0 < b < \frac{1}{2}, \\ \frac{8\alpha_{a,\frac{1}{2}}^2}{(1+2a)\theta^4}, & b = \frac{1}{2}. \end{cases}$$

By (11) and (12), we have

$$\mathrm{Var}\left(F_{i,T}(a,b)\right) = 1 + \varepsilon_{i,T}, \qquad i = 1,2, \tag{22}$$

where

$$\varepsilon_{i,T} = \begin{cases} O(T^{2b-1}), & 0 < b < \frac{1}{2}, \\ O\left((\log T)^{-\frac{1}{2}}\right), & b = \frac{1}{2}. \end{cases} \qquad i = 1,2.$$

The estimates derived by Jiang and Pan in Appendix A[1], together with (11), (12) and Proposition 5, imply that there exist positive constants C_1, C_2, independent of T and y, such that

$$\mathbb{P}\left(\left|F_{2,T}(a,b)\right| \geq y\right) \leq C_1 \exp\{-C_2 y \min\{y, \eta_T\}\}, \tag{23}$$

We apply (23) to the normalized fluctuation of Φ_T. When $0 < b < \frac{1}{2}$, we take $y = Tx$. When $b = \frac{1}{2}$, we take $y = Tx/\sqrt{\log T}$.

Define $G_{i,T}(a,b) = I_1\left(\psi_{i,T}^{a,b}\right)/\left(\sigma_{i,a,b} S_T\right)$, $i = 1,2$. Since $I_1\left(\psi_{i,T}^{a,b}\right)$ is Gaussian,

$$\mathbb{P}\left(\left|G_{i,T}(a,b)\right| \geq x\right) \leq 2\exp\left\{-\frac{x^2 \sigma_{i,a,b}^2 S_T^2}{2 \parallel \psi_{i,T}^{a,b} \parallel_{\mathcal{H}}^2}\right\}, \qquad i = 1,2. \tag{24}$$

Together with (24), this allows the Gaussian tail to be absorbed into the second-order deviation bound after adjusting the constants. The two substitutions and Propositions 1 and 2 then yield (3) and (4).

3.2. Proof of Theorem 1.2

Using the estimates derived by Jiang and Pan in Appendix A[1], for every fixed $\rho > 0$,

$$\sup_{0 \leq x \leq \rho\lambda_T} \left| \log \frac{\mathbb{P}\left(\pm F_{i,T}(a,b) \geq x\sqrt{\mathrm{Var}\left(F_{i,T}(a,b)\right)}\right)}{1 - \Phi(x)} \right| \leq C \frac{(1+\rho\lambda_T)^3}{\eta_T}.$$

The condition on λ_T makes this bound vanish.

Since $\lambda_T^2 \left| \sqrt{\mathrm{Var}\left(F_{i,T}(a,b)\right)} - 1 \right| \leq C\lambda_T^2 \left|\varepsilon_{i,T}\right| \to 0$, the normal-tail ratio under this variance correction tends to one uniformly for $0 \leq x \leq \rho\lambda_T$. Consequently,

$$\sup_{0 \leq x \leq \rho\lambda_T} \left| \frac{\mathbb{P}\left(\pm F_{i,T}(a,b) \geq x\right)}{1 - \Phi(x)} - 1 \right| \to 0, \qquad i = 1,2. \tag{25}$$

By Lemma 4 and the conditions $\lambda_T^3/\eta_T \to 0$ and $a + b > 0$, the Gaussian tail bound for $I_1\left(\psi_{i,T}^{a,b}\right)$ yields

$$\lim_{T \to \infty} \frac{1}{\lambda_T^2} \log \mathbb{P}\left(\left|G_{i,T}(a,b)\right| > \frac{1}{\lambda_T \log \lambda_T}\right) = -\infty, \qquad i = 1,2.$$

Define

$$R_{1,T} = \frac{d_T(a,b) - m_{1,T}(a,b)}{\sigma_{1,a,b} S_T}, \qquad R_{2,T} = \frac{D_T(a,b) - m_{2,T}(a,b)}{\sigma_{2,a,b} S_T}.$$

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From Proposition 2, we have

$$R_{1,T}=\begin{cases}O\left(T^{2b-2-\min\{a,1\}}\log T\right), & 0<b<\frac{1}{2},\\ O\left(T^{-1-\min\{a,1\}}(\log T)^{1/2}\right), & b=\frac{1}{2},\end{cases}\qquad R_{2,T}=\begin{cases}O(T^{2b-1}), & 0<b<\frac{1}{2},\\ O\left((\log T)^{-1/2}\right), & b=\frac{1}{2}.\end{cases}$$

Under the condition on λ_T, these estimates give $|R_{i,T}|\le 1/(\lambda_T\log\lambda_T)$ for $i=1,2$ and all sufficiently large T. By (22), the change in the tail threshold caused by variance normalization is bounded by the same threshold uniformly for $0\le x\le\rho\lambda_T$.

Set $U_{1,T}(a,b)=N_T(a,b), U_{2,T}(a,b)=\Phi_T(a,b)$. By the decompositions in Proposition 1, for $0\le x\le\rho\lambda_T$, and i=1,2,

$$\begin{aligned}&\mathbb{P}\left(\pm F_{i,T}(a,b)\ge x\mp R_{i,T}+\frac{1}{\lambda_T\log\lambda_T}\right)-\mathbb{P}\left(|G_{i,T}(a,b)|>\frac{1}{\lambda_T\log\lambda_T}\right)\\ &\le\mathbb{P}\left(\pm\frac{U_{i,T}(a,b)-m_{i,T}(a,b)}{\sigma_{i,a,b}S_T}\ge x\right)\\ &\le\mathbb{P}\left(\pm F_{i,T}(a,b)\ge x\mp R_{i,T}-\frac{1}{\lambda_T\log\lambda_T}\right)+\mathbb{P}\left(|G_{i,T}(a,b)|>\frac{1}{\lambda_T\log\lambda_T}\right).\end{aligned}\tag{26}$$

Combining (25) with (26) gives the uniform Gaussian tail approximation for the centered and normalized functional for each i=1,2. Substituting the corresponding centering terms and normalization constants yields (5)–(6). This completes the proof.

3.3. Proof of Theorem 1.3

We first consider the standard normalization. From (2), we have

$$\frac{m_{2,T}(a,b)}{\sigma_{1,a,b}S_T}\left(\hat{\theta}_T-\theta\right)=\frac{N_T(a,b)}{\sigma_{1,a,b}S_T}\frac{m_{2,T}(a,b)}{\Phi_T(a,b)}.$$

By (10) and (21), together with the condition $\lambda_T^3/\eta_T\to 0$, this yields

$$\lambda_T\log\lambda_T\left|\frac{m_{1,T}(a,b)}{\sigma_{1,a,b}S_T}\right|\to 0,\qquad\left|\frac{m_{1,T}(a,b)}{\sigma_{1,a,b}S_T}\right|\le\frac{1}{4\lambda_T\log\lambda_T}$$

for T large enough, we obtain

$$\begin{aligned}&\mathbb{P}\left(\left|\frac{m_{2,T}(a,b)}{\sigma_{1,a,b}S_T}\left(\hat{\theta}_T-\theta\right)-\frac{N_T(a,b)-m_{1,T}(a,b)}{\sigma_{1,a,b}S_T}\right|\ge\frac{1}{\lambda_T\log\lambda_T}\right)\\ &\le\mathbb{P}\left(\left|\frac{N_T(a,b)-m_{1,T}(a,b)}{\sigma_{1,a,b}S_T}\right|\ge\lambda_T\log\lambda_T\right)\\ &\quad+\mathbb{P}\left(\left|\frac{\Phi_T(a,b)}{m_{2,T}(a,b)}-1\right|\ge\frac{1}{2}\right)+\mathbb{P}\left(\left|\frac{\Phi_T(a,b)}{m_{2,T}(a,b)}-1\right|\ge\frac{1}{4\lambda_T^2(\log\lambda_T)^2}\right).\end{aligned}\tag{27}$$

By (3), (4), Theorem 1.2, and the condition on λ_T, the right-hand side of (27) is exponentially negligible on the scale λ_T^2. Thus the standard statistic and its centered numerator differ by an exponentially negligible term at the prescribed threshold.

For the self-normalized statistic, we use

$$\frac{\sqrt{m_{2,T}(a,b)\Phi_T(a,b)}}{\sigma_{1,a,b}S_T}\left(\hat{\theta}_T-\theta\right)=\frac{N_T(a,b)}{\sigma_{1,a,b}S_T}\sqrt{\frac{m_{2,T}(a,b)}{\Phi_T(a,b)}}$$

and the elementary bound $\left|\sqrt{m_{2,T}(a,b)/\Phi_T(a,b)}-1\right|\le C\left|\Phi_T(a,b)/m_{2,T}(a,b)-1\right|$ on $\left|\Phi_T(a,b)/m_{2,T}(a,b)-1\right|<1/2$.

Applying the decomposition used in (27) with this local bound gives the same exponential negligibility for the self-normalized statistic at the prescribed threshold. Finally, combining these two exponential-equivalence estimates with Theorem 1.2, and using the stability of the normal tail under the

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shift $1/(\lambda_T \log \lambda_T)$, we obtain (7)–(8). The proof is complete.

4. Conclusions

This paper studies deviation problems for a linear self-attracting diffusion driven by weighted fractional Brownian motion, with emphasis on the tail behavior of the quadratic functionals $N_T(a,b)$, $\Phi_T(a,b)$ and the least squares estimator $\hat{\theta}_T$. Using Wiener–Itô decompositions, these quantities are separated into deterministic leading terms, first-order Wiener integrals and second-order Wiener integrals. The analysis then provides the corresponding deterministic equivalents, kernel norm estimates and contraction bounds.

Based on these estimates, we obtain a deviation inequality for $\Phi_T(a,b)$, which describes the exponential rate at which it deviates from its deterministic scale. We also prove Cramér-type moderate deviations for $N_T(a,b)$ and $\Phi_T(a,b)$, showing that their normalized tail probabilities are asymptotically equivalent to the standard normal tail. These results lead to both standard and self-normalized Cramér-type moderate deviations for $\hat{\theta}_T$. The self-normalized form is useful for statistical inference because it avoids a separate estimation of the asymptotic variance.

The results are obtained for the parameter range $0 < b \leq 1/2$ and $a + b > 0$. Extending the analysis to the full admissible range, especially to the case $b > 1/2$, would require sharper control of kernel norms and contraction terms. It would also be natural to study the joint estimation of (θ, ν), in particular Cramér-type moderate deviations for $\hat{\nu}_T$ and for statistics involving both estimators.

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