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# Middle School Admission in China–A Special Case of Boston Mechanism

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***Abstract:*** *In this paper we study a widely-used mechanism in China to match finite number of primary school graduates to middle schools. The main question is what the equilibrium looks like when the number of schools are small. I solve the equilibrium by first eliminate weakly dominated strategies, then compute the incentive compatibility for each type of students. The equilibrium might or might not be unique, which depends on several variables such as the quota of each schools, the distribution of students' preference and students' valuation of each school available.*

***Keywords:*** *Admission to China; Boston mechanism: equilibrium; Weak domination strategy; Incentive compatibility*

## 1. Introduction

Here is how the mechanism works: We are matching $n$ students to $N$ schools, each with quota $q_i, i = 1,2,\dots, N$ respectively. $\sum_{i=1}^{N} q_i = n$. Every schools are acceptable to each student, so we don't consider student prefer being unmatched than admitted by some schools here. Students first report their preference in forms of an ordinal list of all $N$ schools, with all schools contained and no repetition. Mathematically speaking each student submit a permutation of $N$ schools as their preference. Then computer randomly match these $n$ students with number $1$ to $n$. This unique number assigned to each student represent his place on each school's preference list of students. This means that every schools share a same ranking of students, which completely depends on the random permutation of students by computer. With both sides' preference settled down, Boston Mechanism is then applied.

We first introduce two lemmas which will make our calculation easier in the future.

***Lemma 1.1*** Everyone gets a school.

***Proof of Lemma 1.1*** The proof is trivial. If a student $s$ is not admitted to any school after all $N$ rounds, then the quota of each school must all be fulfilled. This means

$$N \geq \sum_{i=1}^{N} q_i + 1 > \sum_{i=1}^{N} q_i$$

contradict with our general conditions.

***Lemma 1.2*** For any student $i$, if his worst school is $S_j$, then any report with $S_j$ not at the last place is weakly dominated.

***Proof of Lemma 1.2*** For any preference report $P: S_1, S_2, \cdots S_N$ for student $i$, suppose $S_m$ is his worst school. If $m < N$, assume his best school among $S_m, S_{m+1}, \cdots S_N$ is $S_n, m < n \leq N$. We claim that $P'$ generated by switching position of $S_n$ and $S_m$ in $P$ while keeping other school unchanged weakly dominates $P$.

For any fix $P_{-i}$, The expected payoff for s with $P$ and $P'$ is determined by his chance of being admitted in each round as well as his school choice in that round. To be more precise,

$$E_i(P, P_{-i}) = p_1 \cdot V_i(S_1) + (1 - p_1)p_2 \cdot V_i(S_2) + \cdots + (1 - p_1)(1 - p_2) \cdots (1 - p_{n-1})p_n \cdot V_i(S_n)$$

$$E_i(P', P_{-i}) = p'_1 \cdot V_i(S_1) + (1 - p_1)p_2 \cdot V_i(S_2) + \cdots + (1 - p_1)(1 - p_2) \cdots (1 - p_{n-1})p_n \cdot V_i(S_n)$$

Here $p_i(p'_i)$ stands for the probability that when reaching the $i$-th round, his probability of being admitted in that round. By defination of $P'$, $p'_i = p_i$, $P_i = P'_i$ for $i \neq m, n$.

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Thus we have

$$E_i(P', P_{-i}) - E_i(P, P_{-i}) = \underbrace{(q'_m \cdot V_i(S_n) + q'_{m+1} \cdot V_i(S_{m+1}) + \cdots + q_{n'} \cdot V_i(S_m) + \cdots + q'_N \cdot V_i(S'_N))}_{A}$$

$$- \underbrace{(q_m \cdot V_i(S_m) + q_{m+1} \cdot V_i(P_{S+1}) + \cdots + q_n \cdot V_i(S_n) + \cdots + q_N \cdot V_i(S_N))}_{B}$$

where $q'_j = \prod_{i=1}^{j-1} (1 - p'_i)p'_j$ and $q_j = \prod_{i=1}^{j-1} (1 - p_i)p_j$ for $j = m, m+1, \cdots, N$.

Notice that $\sum_{j=m}^{N} q'_j = \sum_{j=m}^{N} q_j$ because $P$ and $P'$ are same among the first $m-1$ rounds. We can consider A and B as two weighted average of $V_i(S_k), k = m, m+1, \cdots, N$ with total weights $q = \sum_{j=m}^{N} q'_j$.

$p'_m = 1$: $A = q \cdot V_i(S_n)$ reaches the maximal value since all the weights are given to the largest term $V_i(S_n)$.

$p_m = 1$: $B = q \cdot V_i(S_m)$ reaches the minimal value since all the weights are given to the smallest term $V_i(S_m)$.

$p'_m \leq p_m < 1$: $p_n = p'_n = 0$, $q'_n \leq q_m, q'_m \geq q_n, q'_i \geq q_i$ for $i \neq m, n$. Compared to B, A reduce(or keep) the weights of the smallest term and increase (or keep) the weights of all other terms. So we must have $A \geq B$.

$p_m \leq p'_m < 1$: $p_n = p'_n = 0$, $q'_n \leq q_m, q'_m \geq q_n, q'_i \leq q_i$ for $i \neq m, n$. Compared to B, A remove(or keep) the weights of all other terms to (or keep) $V_i(S_n)$, which is the largest term. So again we must have $A \geq B$.

## 2. Equilibrium

In this section we calculate the equilibrium for small N. We first calrify some notations and words. A student's "best" school is the school on the top of his true preference list, while the "worst" school is the school at the least of his true preference list. We might use $[S_1, S_2, S_3]$ or equivalent notation for short of preference $[S_1 > S_2 > S_3]$ or equivalent. We might use the word "honest" for students who truthfully report their preference, and the word "lying" for students who manipulate. These are just for convenience and has nothing to do with moral judgement.

***Assumption 2.1.*** We assume that each student's payoff for entering his best school are the same: $V_1$, payoff for entering 2nd best school are the same: $V_2$, etc. $V_i > V_j$ for $i < j$.

We understand that this assumption might not be true in some special cases, for instance some students might not differ between schools as other students do, or some students might be in-different between some schools. However, in reality, while choosing middle school to enter, students and their family usually lives on a ordinal preferences of schools, but not a precise cardinal order. By such assumption we can simplify our model quantitatively without qualitative sacrifice.

N = 2

Suppose there are only 2 schools $S_1$ and $S_2$ for n students to choose, with quota $q_1$ and $q_2$ respectively. In this case, everyone truth reporting is an equilibrium by Lemma 1.2: Diviating from true preference leads to ranking one's worst school among $s_1$ and $s_2$ not at second place(the last place).

We claim this is the only equilibrium, to be more specific, truth reporting is a dominate strategy when $N = 2$.

Proof of claim: Consider an arbitrary student i. Assume i's true preference is $P_i = [S_1 > S_2]$ without lost of generality. Suppose besides i, there are a students reporting $P = (S_1, S_2)$, b students reporting $P = (S_2, S_1)$, with $a, b \geq 0$ and $a + b = n - 1$. We then compute the expected payoff of student i:

(i) If $a \leq q_1 - 1$,then $b = q_1 + q_2 - 1 - a \geq q_2$. We have

$$E_i(P_i, P_{-i}) - E_i(P'_i = (S_2, S_1), P_{-i}) = V_1 - (\frac{q_2}{b+1} V_2 + (1 - \frac{q_2}{b+1}) V_1) = \frac{q_2}{b+1} (V_1 - V_2) > 0$$

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Noted that $a \leq q_1 - 1$, so $i$ will certainly be admitted to $S_1$ in the first round with $P_i$.

(ii) If $a \geq q_1$,then $b = q_1 + q_2 - 1 - a \leq q_2 - 1$. We have

$E_i(P_i, P_{-i}) - E_i(P_i' = (S_2, S_1), P_{-i}) = (\frac{q_1}{a+1} V_1 + (1 - \frac{q_1}{a+1}) V_2) - V_2 = \frac{q_1}{a+1}(V_1 - V_2) > 0$

Noted that $b \leq q_2 - 1$, so $i$ will certainly be admitted to $S_2$ in the first round with $P_{-i}$.

This shows that student with preference $S_1 > S_2$ should always truthfully report their preference. The proof for student with $S_2 > S_1$ is similar.

Truth-reporting is a strictly dominating strategy for all. Everyone will truthfully report their preference when there are only two schools available.

N = 3

Suppose now we have 3 schools $S_1, S_2, S_3$ available, with quota $q_i, i = 1,2,3$ respectively. The calculation is much more complicated then the previous and we will start with special cases.

### *2.1 Common Preference*

We first discuss the simplest case where every students have the same preference profile. Without lost of generality suppose that everyone likes $S_1$ better than $S_2$ better than $S_3$, which means $P_i = [S_1 > S_2 > S_3]$ for all i. For arbitrary student i, by Lemma 1.2, we only need to consider one deviation: $P_i' = [S_2 > S_1 > S_3]$. Suppose in equilibrium there are $t_1$ students reporting $[S_1, S_2, S_3]$, $t_2$ students reporting $[S_2, S_1, S_3]$. If $t_2 \geq q_2$, we call it a Type 1 equilibrium. If $0 < t_2 < q_2$, we call it a Type 2 equilibrium.

***Proposition 2.1***. $t_1 \geq q_1$

Proof: The proof is trivial. If $t_1 < q_1$,those who deiviated from truth preference should return to their truth preference(given other student's report fixed) because he could guarantee a place in $S_1$ by doing so. Thus it can't be an equilibrium.

***Proposition*** 2.2. If $q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3 \neq n \cdot V_2$, then Type 2 equilibrium doesn't exist.

Proof: Proof by contradiction. If $q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3 > n \cdot V_2$:

Suppose there is an equilibrium such that $0 < t_2 < q_2$, then $t_1 = n - t_2 > q_1$. For these $t_2$ students with $P' = [S_2, S_1, S_3]$, their incentive compatibility gives:

$$E[P', P_{-i}] = V_2 \geq \frac{q_1}{t_1 + 1} \cdot V_1 + (1 - \frac{q_1}{t_1 + 1}) \cdot (\frac{q_2 - t_2 + 1}{t_1 + 1 - q_1} \cdot V_2 + (1 - \frac{q_2 - t_2 + 1}{t_1 + 1 - q_1}) \cdot V_3) = E[P, P_{-i}]$$

Notice that $(1 - \frac{q_2 - t_2 + 1}{t_1 + 1 - q_1}) = \frac{t_1 + 1 - q_1 - q_2 + t_2 - 1}{t_1 + 1 - q_1} = \frac{q_3}{t_1 + 1 - q_1}$, thus we have

$V_2 \geq \frac{q_1}{t_1 + 1} \cdot V_1 + \frac{q_2 - t_2 + 1}{t_1 + 1} \cdot V_2 + \frac{q_3}{t_1 + 1} \cdot V_3$

Multiply both side with $t + 1$ and add $(t_2 - 1) \cdot V_2$:

$(t_1 + t_2) \cdot V_2 \geq q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3$

$t_1 + t_2 = n$, so this contradict with our assumption. In this case everyone truthfully report is an equilibrium and $t_2 = 0$

Similarly, if $q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3 < n \cdot V_2$:

Consider incentive compatibility of a truth reporting student:

$$E[P, P_{-i}] = \frac{q_1}{t_1} \cdot V_1 + (1 - \frac{q_1}{t_1}) \cdot (\frac{q_2 - t_2}{t_1 - q_1} \cdot V_2 + (1 - \frac{q_2 - t_2}{t_1 - q_1}) \cdot V_3) \geq V_2 = E[P, P_{-i}]$$

Since $(1 - \frac{q_2 - t_2}{t_1 - q_1}) = \frac{t_1 - q_1 - q_2 + t_2}{t_1 - q_1} = \frac{q_3}{t_1 - q_1}$,we have

$\frac{q_1}{t_1} \cdot V_1 + \frac{q_2 - t_2}{t_1} \cdot V_2 + \frac{q_3}{t_1} \cdot V_3 \geq V_2$

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which is equivalent to

$$q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3 \geq (t_1 + t_2) \cdot V_2$$

This again contradict with our assumption.

This proposition shows that only if $q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3 = n \cdot V_2$ should we consider Type 2 equilibrium. More importantly, we can show that every Type 2 equilibrium yields same expected payoff.

***Corollary 2.3.*** If $q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3 = n \cdot V_2$, then $t_2 = 0,1,2,\cdots,q_2 - 1$ each gives an equilibrium. The expected payoff for each student equals $V_2$ in all these equilibriums.

Proof: We first proof the case of $t_2 = 0$. Expected payoff for any student given that everyone truthfully report $P = [S_1, S_2, S_3]$ is

$E[P, P_{-i}] = \frac{q_1}{t_1} \cdot V_1 + (1 - \frac{q_1}{t_1}) \cdot (\frac{q_2}{t_1 - q_1} \cdot V_2 + (1 - \frac{q_2}{t_1 - q_1}) \cdot V_3)$

Notice that $t_1 = n$, $1 - \frac{q_2}{t_1 - q_1} = \frac{t_1 - q_1 - q_2}{t_1 - q_1} = \frac{q_3}{t_1 - q_1}$, so

$E[P, P_{-i}] = \frac{1}{n} \cdot (q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3) = V_2$

Deviating to $P' = [S_2, S_1, S_3]$ while others report truthfully also yields payoff of $V_2$, so everyone truth reporting $(t_2 = 0)$ is indeed an equilibrium. Proof for other values of $t_2$ is basically the same as what we do in Proposition 2.2, hence omitted here.

We then discuss Type 1 equilibrium. Suppose $(t_1, t_2)$ leads to a Type 1 equilibrium. Then for those who truthfully reports $P = [S_1, S_2, S_3]$, since $t_2 > q_2$, which means all the quota of $S_2$ will be given out in the first round, so these students with $P$ will definately not be admitted by $S_2$. Similarly, those with $P' = [S_2, S_1, S_3]$ will not be admitted by $S_1$ due to the assumption that $t_1 > q_1$. For $t_1$ truth reporting students, we have

$$E[P, P_{-i}] = \frac{q_1}{t_1} \cdot V_1 + (1 - \frac{q_1}{t_1}) \cdot V_3 \geq \frac{q_2}{t_2 + 1} \cdot V_2 + (1 - \frac{q_2}{t_2 + 1}) \cdot V_3 = E[P', P_{-i}]$$

which can be simplified to

$$(t_2 + 1)q_1 \cdot V_1 - (t_2 + 1)q_1 \cdot V_3 \geq t_1 q_2 \cdot V_2 - t_1 q_2 \cdot V_3$$

Recall that $t_2 = n - t_1$, thus we have

$$(n + 1)q_1 \cdot (V_1 - V_3) \geq t_1 q_1 \cdot (V_1 - V_3) + t_1 q_2 \cdot (V_2 - V_3)$$

Notice $q_1 \cdot (V_1 - V_3) + q_2 \cdot (V_2 - V_3) > 0$, hence

$$t_1 \leq \frac{(n + 1)q_1 \cdot (V_1 - V_3)}{q_1 \cdot (V_1 - V_3) + q_2 \cdot (V_2 - V_3)} = U$$

For the lower bound of $t_1$, consider $t_2$ "lying" students. For them

$$E[P', P_{-i}] = \frac{q_2}{t_2} \cdot V_2 + (1 - \frac{q_2}{t_2}) \cdot V_3 \geq \frac{q_1}{t_1 + 1} \cdot V_1 + (1 - \frac{q_1}{t_1 + 1}) \cdot V_3 = E[P, P_{-i}]$$

Following the same process we similarly get

$$t_2 \leq \frac{(n + 1)q_2 \cdot (V_2 - V_3)}{q_2 \cdot (V_2 - V_3) + q_1 \cdot (V_1 - V_3)}$$

so the lower bound of $t_1$ is given by

$$t_1 = n - t_2 \geq \frac{nq_1 \cdot (V_1 - V_3) + nq_2 \cdot (V_2 - V_3) - (n + 1)q_2 \cdot (V_2 - V_3)}{q_2 \cdot (V_2 - V_3) + q_1 \cdot (V_1 - V_3)}$$
$$= \frac{nq_1 \cdot (V_1 - V_3) - q_2 \cdot (V_2 - V_3)}{q_2 \cdot (V_2 - V_3) + q_1 \cdot (V_1 - V_3)} = L$$

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Observe that

$$U - L = \frac{(n+1)q_1 \cdot (V_1 - V_3) - nq_1 \cdot (V_1 - V_3) - q_2 \cdot (V_2 - V_3)}{q_2 \cdot (V_2 - V_3) + q_1 \cdot (V_1 - V_3)} = 1$$

$U \geq t_1 \geq L$, so if U(or L) is not an integer, then there is an unique integer between U and L, which gives us the only solution for $t_1$. If U and L are integers, then we have two solutions for $t_1$.

### *2.2 Common Dislike*

We now move on to the case where there are two preference types among all the students. What's special here is that these two preference list the same school at the lowest place. The reason we care about this situation is a direct result of Lemma 1.2: Rational students will not consider weakly dominated reports. They will only consider deviating from true preference to those which also list their worst school at the bottom.

Suppose there are $k_1$ and $k_2$ students with true preference $P_1 = [S_1, S_2, S_3]$ and $P_2 = [S_2, S_1, S_3]$ respectively. $k_1 + k_2 = n$. There are three potential cases:

- $k_1 \geq q_1, k_2 \geq q_2$
- $k_1 \geq q_1, k_2 < q_2$
- $k_1 < q_1, k_2 \geq q_2$

We primarily focus on the first case since the last two cases are symmetric and could be solved similarly by existing result.

Given that $k_1 \geq q_1, k_2 \geq q_2$, it's clear that Type 2 equilibrium doesn't exist: If the quota of a school is not fulfilled at the first round, then there must exist some students who like this school best but didn't list it at the top. These students then have incentive to deviate(to the report which lists this school at the first place).

As for Type 1 equilibrium, suppose $(t_1, t_2)$ leads to a Type 1 equilibrium. There are at most 4 kinds of students:

1) Honest student with true preference $P_1$ and report $P_1$

2) Honest student with true preference $P_2$ and report $P_2$

3) Lying student with true preference $P_1$ and report $P_2$

4) Lying student with true preference $P_2$ and report $P_1$

We can then calculate $t_1$ and $t_2$ using the incentive compatibility of these four kinds of student.

For honest student with true preference $P_1$, we have:

$E[P_1, P_{-i}] = \frac{q_1}{t_1} \cdot V_1 + (1 - \frac{q_1}{t_1}) \cdot V_3 \geq \frac{q_2}{t_2+1} \cdot V_2 + (1 - \frac{q_2}{t_2+1}) \cdot V_3 = E[P_1', P_{-i}]$

By previous result we know the solution of $t_1$ is given by (2.1.2):

$t_1 \leq \frac{(n+1)q_1 \cdot (V_1 - V_3)}{q_1 \cdot (V_1 - V_3) + q_2 \cdot (V_2 - V_3)} = U_1$

Similarly, for lying student with true preference $P_1$:

$E[P_1', P_{-i}] = \frac{q_2}{t_2} \cdot V_2 + (1 - \frac{q_2}{t_2}) \cdot V_3 \geq \frac{q_1}{t_1+1} \cdot V_1 + (1 - \frac{q_1}{t_1+1}) \cdot V_3 = E[P_1, P_{-i}]$

By (2.1.3), we have

$t_1 \geq \frac{nq_1 \cdot (V_1 - V_3) - q_2 \cdot (V_2 - V_3)}{q_1 \cdot (V_1 - V_3) + q_2 \cdot (V_2 - V_3)} = L_1$

For honest student with preference $P_2$, we have

$E[P_2, P_{-i}] = \frac{q_2}{t_2} \cdot V_1 + (1 - \frac{q_2}{t_2}) \cdot V_3 \geq \frac{q_1}{t_1+1} \cdot V_2 + (1 - \frac{q_1}{t_1+1}) \cdot V_3 = E[P_2', P_{-i}]$

which simply gives

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$t_2 \le \frac{(n+1)q_2\cdot(V_1-V_3)}{q_2\cdot(V_1-V_3)+q_1\cdot(V_2-V_3)}$

Thus we have

$t_1 = n - t_2 \ge n - \frac{(n+1)q_2\cdot(V_1-V_3)}{q_2\cdot(V_1-V_3)+q_1\cdot(V_2-V_3)} = \frac{nq_1\cdot(V_2-V_3)-q_2\cdot(V_1-V_3)}{q_1\cdot(V_2-V_3)+q_2\cdot(V_1-V_3)} = L_2$

Similarly the incentive compatibility of lying student with preference $P_2$ gives

$t_1 \le \frac{(n+1)q_1\cdot(V_2-V_3)}{q_1\cdot(V_2-V_3)+q_2\cdot(V_1-V_3)} = U_2$

Notice that $L_2 < L_1$, $U_2 < U_1$ and $U_1 - L_1 = 1$, $U_2 - L_2 = 1$.

Calculation above directly leads to the following result:

*Proposition 2.3.*

• If $k_1 > U_1$, then all students with preference $P_2$ truthfully report while $[L_1, U_1]$ students with preference $P_1$ truthfully report and $k_1 - [L_1, U_1]$ students with preference $P_1$ manipulate is an equilibrium.

• If $U_1 \ge k_1 \ge L_2$, then everyone truthfully report is an equilibrium.

• If $L_2 > k_1$, then all students with preference $P_1$ truthfully report while $n - [L_2, U_2]$ students with preference $P_2$ truthfully report and $[L_2, U_2] - k_1$ students with preference $P_2$ manipulate is an equilibrium.

We can apply this proposition to the last 2 cases directly.

### *2.3 Common Best, Different Worst*

There is one more special case we need to settle down before we move on to the most general case. Suppose again we have 2 types of preference among all the students: $P_1 = [S_1, S_2, S_3]$ and $P_2 = [S_1, S_3, S_2]$. Notice that in Common Dislike case, deviation from $P_1$ is exactly $P_2$, so in all there are only 2 preferences involved. But here since $P_1' = [S_2, S_1, S_3] \ne P_2$, $P_2' = [S_3, S_1, S_2] \ne P_1$, hence we have to consider 4 different reports in equilibrium.

Suppose there are $k_{1,2}$ students with true preference $P_{1,2}$, respectively, $k_1 + k_2 = n$.Suppose that $t_1$ students report $P_1$, $t_2$ students report $P_2$, $t_3$ students report $P_1'$ and $t_4$ students report $P_2'$ leads to an equilibrium, $t_1 + t_3 = k_1$,$t_2 + t_4 = k_2$. We must have $t_1 + t_2 \ge q_1$, hence $t_3 + t_4 \le q_2 + q_3$.

Notice that If $k_1 \ge q_1 + q_2$, which means number of students who report $P_1$ or $P_1'$ are larger than the sum of quota of $S_1$ and $S_2$, then $S_2$ must be fulfilled in the first two rounds since $P_1$ and $P_1'$ both have $S_2$ in the first two places. This tells us that other students will never be admitted by $S_2$ if they put $S_2$ at the last place. By lemma 1.1 we immediately know that student who report $P_2$ and $P_2'$ will be admitted in the first two rounds. We thus get the following result by lemma 1.2:

***Proposition 2.4.*** If $k_1 \ge q_1 + q_2$, then students with true preference $P_2$ should truthfully report their preference.We have $t_2 = k_2, t_4 = 0$ Symmetrically, if $k_2 \ge q_1 + q_3$, then students with true preference $P_1$ should truthfully report their preference. We have $t_1 = k_1, t_3 = 0$.

Now, suppose $k_1 \ge q_1 + q_2, k_2 < q_3$. With $k_2$ students truthfully report $P_2$, we can apply what we did in Common Preference model to solve for the equilibrium. Assume $t_1$ students report $P_1$, $t_3$ students report $P_1'$ leads to an equilibrium, where $t_1 + t_3 = k_1 = n - k_2 > q_1 + q_2$, $t_1 + k_2 \ge q_1$. Then for these $t_1$ students, if $t_3 < q_2$

$$\frac{q_1}{t_1+k_2}\cdot V_1 + \left(1-\frac{q_1}{t_1+k_2}\right)\left(\frac{q_2-t_3}{t_1-t_1\cdot\frac{q_1}{t_1+k_2}}\cdot V_2 + \left(1-\frac{q_2-t_3}{t_1-t_1\cdot\frac{q_1}{t_1+k_2}}\right)\cdot V_3\right) \ge V_2 = E[P_1', P_{-i}]$$

Notice that if we let $t_1' = t_1, q'_1 = t_1 \cdot \frac{q_1}{t_1+k_2}$, the inequality above has exactly the same form of Inequality (2.1), similarly for the incentive compatibility for $t_3$ students who report $P_1'$. By corollary 2.3, such (type 2) equilibrium doesn't exist.

If $t_2 \ge q_2$, for $t_1$ honest students, we have

$$\frac{q_1}{t_1+k_2}\cdot V_1 + \left(1-\frac{q_1}{t_1+k_2}\right)\cdot V_3 \ge \frac{q_2}{t_2+1}\cdot V_2 + \left(1-\frac{q_2}{t_2+1}\right)\cdot V_3$$

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Let $t_1' = t_1 + k_2$, this inequality is the same as what we solved in Common Preference case, similarly for the $t_3$ lying students. The calculation is basically the same hence omitted here.

Now we consider $k_1 < q_1 + q_2$, $k_2 < q_1 + q_3$. If $t_3 \geq q_2$, which means $S_2$ will be full after round 1, hence students with true preference $P_2$ should truthfully report their preference. We then have $t_2 = k_2$ and again we are back to Common Preference case, symmetrically for $t_4 \geq q_3$. Our last concern is the case where $t_3 < q_2, q_2 < t_1 + t_3 < q_1 + q_2$ and symmetrically $t_4 < q_3, q_3 < t_2 + t_4 < q_1 + q_3$.

First assume $t_1 > q_1$, then for $t_1$ students who reports $P_1$, their chance $Pro_1$ for being admitted to $S_1$ is $\frac{q_1}{t_1+t_2}$.

We then calculate their chance $Pro_2$ for being admitted to $S_2$. Fix a $P_1$ student s(short for students who report $P_1$ in equilibrium). For $0 \leq i \leq q_1$, the probability $A_i$ that exactly i $P_1$ students (but not s) are admitted to $S_1$ is given by

$$A_i^{t_1} = \frac{\binom{t_1 - 1}{i} \cdot \binom{t_2}{q_1 - i}}{\binom{t_1 + t_2}{q_1}}$$

let $x = t_1 + t_3 - q_2$. For $i \leq x$, If s was not admitted in the first round, the probability $B_i$ that s are admitted to $S_2$ is then given by

$$B_i^{t_1} = \frac{q_2 - (k_1 - t_1)}{t_1 - i}$$

For $x < i \leq q_1$, notice that if no less than x $P_1$ students are admitted to $S_1$, the remaining $P_1$ students will be garunteed a place in $S_2$ at round 2, we have $B_i^{t_1} = 1$ since $q_2 - t_3 > t_1 - i$. We claim that

$$Pro_2 = \sum_{i=0}^{q_1} (A_i^{t_1} \cdot B_i^{t_1})$$

Then consider the incentive compatibility for student s:

$$E[P_1, P_{-s}] = Pro_1 \cdot V_1 + Pro_2 \cdot V_2 + (1 - Pro_1 - Pro_2) \cdot V_3 \geq V_2 = E[P_1', P_S]$$

For $t_2$ students who report $P_1'$, their incentive compatibility is given by:

$$E[P_1', P_{-i}] = V_2 \geq Pro_1' \cdot V_1 + Pro_2' \cdot V_2 + (1 - Pro_1' - Pro_2') \cdot V_3 = E[P_1, P_{-i}]$$

Here $Pro_1' = \frac{q_1}{t_1+1+t_2}$, and $Pro_2' = \sum_{i=0}^{q_1} A_i^{t_1+1} \cdot B_i^{t_1+1}$. Solving this two inequality we will get the upper and lower bound of $t_1$ in forms of a function of $t_2$:

$$F(k_1, k_2, t_2) \geq t_1 \geq G(k_1, k_2, t_2)$$

Remark: for $t_1 \leq q_1$, we have $Pro_2 = \sum_{i=0}^{t_1-1} (A_i^{t_1} \cdot B_i^{t_i})$, which allow us to continue with similar calculation. Using the same method to analyze $t_2$, we will get

$$H(k_1, k_2, t_1) \geq t_2 \geq I(k_1, k_2, t_1)$$

Combining (2.3.3) and (2.3.4) we can solve for $t_1$ and $t_3$.

### *2.4 Generalization*

Now we are ready to solve the most general cases, where we consider all 6 preferences among students. We use $n(S_i)$ to denote the number of students who like $S_i$ best. Since $n(S_1) + n(S_2) + n(S_3) = n = q_1 + q_2 + q_3$, we have at least one $i = 1,2$ or 3 such that $S_i \geq q_i$. Without lost of generality we assume that $n(S_1) \geq q_1$.

(i): $n(S_1) \geq q_1, n(S_2) < q_2, n(S_3) < q_3$. Given that $n(S_1) \geq q_1$, in equilibrium we must have the

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quota of $S_1$ fulfilled in the first round. Thus for students with true preference $[S_2, S_3, S_1]$ and $[S_3, S_2, S_1]$, they don't need to worried about getting into their worst school $S_1$ because they will certainly be admitted in the first two rounds. Hence by Lemma 2.1, truthfully report their preference is a weakly dominating strategy. We call the group of these students $D_1$, short for "don't like $S_1$"

Divide the remaining four types of students into two groups: $D_2$ ={students with true preference $[S_1, S_3, S_2]$ or $[S_3, S_1, S_2]$} and $D_3$ ={students with true preference $[S_1, S_2, S_3]$ or $[S_2, S_1, S_3]$}. We can then apply our analysis in previous case to solve the equilibrium for these case. Remark: $|D_2|, |D_3|$ are fixed, which can be used as $k_{1,2}$ as in prevous case.

(ii): $n(S_1) \geq q_1, n(S_2) \geq q_2, n(S_3) < q_3$. Similar to previous case, in equilibrium we must have the quota of $S_1$ and $S_2$ fulfilled in the first round. We claim that students of type $[S_1, S_3, S_2], [S_3, S_1, S_2], [S_2, S_3, S_1]$ and $[S_3, S_2, S_1]$ should all truthfully report their type as a direct result of lemma 2.1. The game could them be simplifed to the Common Dislike case between students with true preference $P_1 = [S_1, S_2, S_3]$ and $P_2 = [S_2, S_1, S_3]$.

Suppose in equilibrium there are $t_1$ students report $P_1$ and $t_2$ students report $P_2$, then for honest students with true preference $P_1$ and report $P_1$, we have

$$\frac{q_1}{t_1 + x} \cdot V_1 + (1 - \frac{q_1}{t_1 + x}) \cdot V_3 \geq \frac{q_2}{t_2 + y + 1} \cdot V_2 + (1 - \frac{q_2}{t_2 + y + 1}) \cdot V_3$$

Here $x$ is the number of students with true preference $[S_1, S_3, S_2]$, and $y$ is the number of students with true preference $[S_2, S_3, S_1]$. Notice that $t_1 + t_2$ equals the number of students who don't like $S_3$ most, and $x, y$ are constants, hence this linear inequation is similar to the one we saw in Common Dislike case. Moreover, the incentive compatibility of other 3 kinds of students, namely honest students with true preference $P_2$, lying students with true prference $P_1$ and $P_2$ respectively also involve only $t_1, t_2$ and $x, y$. The system we get is exactly a system of Common Dislike which we already solved, despite the fact that numbers are different.

(iii): $n(S_1) \geq q_1, n(S_2) \geq q_2, n(S_3) \geq q_3$. In this case we must have $n(S_1) = q_i, i = 1,2,3$. Everyone truthfully report their preference and all students get into their best schools.

Students' objective to deviate from their true preference mainly consist of aversion of risk. For an arbitrary student $s$, if large number of others like his worst school, then his chance for entering that school is low. He thus has less incentive to deviate. On the contrary, if relatively less students like his worst school(or lots of students like his best school), in order to avoid getting into it, he might give up his best school to seek a position in the second best.

## 3. Extension

In this section we introduce another silimar mechanism and make a simple comparison between these two for special cases. The difference between these two mechanisms is really simple: In our new mechanism, students report their preferences after they were assigned with ranking numbers. We are interested in how this slight change affect students' strategies as well as social welfare. We called our previous mechanism $M_1$ and the new one $M_2$ for short.

### *3.1 Common Preference*

Suppose all students share common preference $[S_1, S_2, S_3]$. Fix a student $s$, assume he got number x, with $1 \leq x \leq n$.

(i). If $1 \leq x \leq q_1$, then $s$ should truthfully report $S_1$ as his best school because such a small $x$ will garuntee his admission by $S_1$.

(ii). If $q_1 < x \leq q_1 + q_2$, there will be no place for $s$ in $S_1$ even though he list $S_1$ as his best school. He thus should list $S_2$ at top of his preference, since $S_2$ is his best option left and his number $x$ will garuntee his admission by $S_2$.

(iii). If $q_1 + q_2 < x \leq n$, similarly there will be no place for $s$ in both $S_1$ and $S_2$, he will then be admitted by $S_3$, regardless of what he reports.

We then see that the maximal expected return for $x$ is given by

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$$E_{M_2} = \frac{q_1}{n} \cdot V_1 + \frac{q_2}{n} \cdot V_2 + \frac{q_3}{n} \cdot V_3$$

We see that strategy "Report $[S_1, S_2, S_3]$ if $x \le q_1$, Report $[S_2, S_1, S_3]$ if $q_1 < x$" is a weakly dominating strategy for every students, and they all can reach $E_{M_2}$ by doing so.

Recall that in the Common Preference case for $M_1$, if $q_1 \cdot V_1 + q_2 \cdot V_2 + q_3 \cdot V_3 \ge n \cdot V_2$, every students will truthfully report, which leads to a expected return

$$E_{M_1} = \frac{q_1}{n} \cdot V_1 + \frac{q_2}{n} \cdot V_2 + \frac{q_3}{n} \cdot V_3$$

for all students. We then have $E_{M_1} = E_{M_2}$.

If $\frac{q_1}{n} \cdot V_1 + \frac{q_2}{n} \cdot V_2 + \frac{q_3}{n} \cdot V_3 < n \cdot V_2$, recall that we calculated $U - 1 \le t_1 \le U$ in (2.1.2) and (2.1.3). Hence for those who truthfully report $[S_1, S_2, S_3]$, we have

$$\frac{q_1}{U} \cdot V_1 + (1 - \frac{q_1}{U}) \cdot V_3 \le E_{M_1} \le \frac{q_1}{U-1} \cdot V_1 + (1 - \frac{q_1}{U-1}) \cdot V_3$$

Let $V_1 = \alpha \cdot V_3$, $V_2 = \beta \cdot V_3$, $\alpha > \beta > 1$. Notice that $\frac{q_1}{U} \cdot V_1 + (1 - \frac{q_1}{U}) \cdot V_3 \ge E_{M_2}$ implies

$$\frac{nq_1(\alpha - 1)}{q_1\alpha + q_2\beta + q_3 - n} \ge U$$

If $U$ satisfy (3.1.1), then $E_{M_2} < E_{M_1}$ for honest students in $M_1$. Similarly, $\frac{q_1}{U-1} \cdot V_1 + (1 - \frac{q_1}{U-1}) \cdot V_3 \le E_{M_2}$ implies

$$\frac{nq_1(\alpha - 1)}{q_1\alpha + q_2\beta + q_3 - n} + 1 \le U$$

If $U$ satisfy (3.1.2), then $E_{M_2} > E_{M_1}$ for honest students in $M_1$. We can similarly calculte the expected return for lying students. Combining these interval provides us with a range in which we can say that the equilibrium of one mechanism is more efficient than the equilibrium of the other.

## 4. Conclusion

We calculated the equilibrium under different distribution of students' preferences. When student's payoff for entering their best school is relatively larger, more students will truthfully report and take the risk of being admitted by their worst school. The equilibrium might not be unique, however they are qualitatively identical and only differ in numbers. Letting students know their ranks before submitting their preference might increase social welfare, by not for sure. The comparison and refinement of this mechanism is worth to be further discussed.


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