Complete Convergence for Weighted Sums of Negatively Dependent Random Variables under Sub-linear Expectations

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Abstract: In this article, we study complete convergence theorems for weighted sums of negatively dependent random variables under the sub-linear expectations. Our results extend the corresponding results of complete convergence in the classical probability.

Keywords: Complete convergence, Negatively dependent, Random variables, Sub-linear expectations.

1. Introduction

The complete convergence is a very important research field in probability limit theory. For classical probability space, it was first introduced by Hsu and Robbins in 1947 as follows: A sequence \( \{X_n, n \geq 1\} \) of random variables converges completely to a constant \( C \) if \( \sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty \) for all \( \varepsilon > 0 \). After that, many scholars have obtained the complete convergence on other random variable sequences, for example, Wu qun-ying obtained the complete convergence for a sequence of independent and identically distributed random variables. Wu and Jiang et al. gave the complete convergence and complete moment convergence for negatively dependent random variables sequence.

In the classical probability space, the additivity of the probabilities and the expectations are assumed. But in practice, such additivity assumption is not feasible in many areas of applications because the uncertainty phenomena cannot be modeled using additive probabilities or additive expectations. Therefore in recent years, some scholars began to extend the complete convergence of classical probability theory to sub-linear space, for example, Ling obtained complete convergence of random variables sequence under sub-linear expectation space, Feng researched complete convergence for weighted sums of negatively dependent random variables, Yu and Wu researched the Marcinkiewicz type complete convergence for weighted sums under sub-linear expectations.

Complete convergence for weighted sums are also important in sublinear expectation space, which can be applied to nonparametric regression models. The main purpose of this article is to establish complete convergence theorems for arrays of rowwise negatively dependent random variables under the sub-linear expectations.

2. Symbol Description

(1) \( a_n \ll b_n \) denote that for sufficiently large \( n \), there exist \( c > 0 \) such that \( a_n \leq cb_n \);

(2) \( I(\cdot) \) denote an indicator function;

(3) \( \log(x) = \max(1, \ln x) \).

3. The Basic Definition and Properties of Sub-linear Expectation Space

We use the framework and notation of sub-linear expectation established by Peng.

Let \( (\Omega, F) \) be a given measurable space, \( H \) be a linear space of real function defined on \( (\Omega, F) \), such that if \( X_1, X_2, \ldots, X_n \in H \), then \( \phi(X_1, X_2, \ldots, X_n) \in H \) for each
\[ \varphi \in C_{1,1,q}(R^n), \text{ where } C_{1,1,q}(R^n) \text{ denotes the linear space of local Lipschitz functions } \varphi \text{ satisfying} \]
\[ |\varphi(x) - \varphi(y)| \leq c(|x| + |y|)^m |x - y|, \forall x, y \in R^n \]

For some \( C > 0, m \in \mathbb{N} \) depending on \( \varphi \). \( H \) is considered as a space of “random variables”. If \( X \) is an element of set \( H \), then we denote \( X \in H \).

**Definition 1** A sub-linear expectation \( E \) is a function \( E: H \to R \) satisfying the following properties: for all \( X, Y \in H \), we have

(a) Monotonicity: if \( X \geq Y \) then \( EX \geq EY \);

(b) Constant preserving: \( Ec = c \), for all \( c \in R \);

(c) Sub-additivity: \( E(X + Y) \leq EX + EY \) whenever \( EX + EY \) is not of the form \( + \infty - \infty \) or \( - \infty + \infty \);

(d) Positive homogeneity: \( E(\lambda X) = \lambda EX, \lambda > 0 \).

\( E \) is called sub-linear expectation on \( H \). Here \( R = [-\infty, +\infty] \). The triple \( (\Omega, H, E) \) is called a sub-linear expectation space. Give a sub-linear expectation \( E \), let us denote the conjugate expectation \( E^* \) of \( E \) by \( E^*X = -E(-X) \), \( \forall X \in H \).

From the definition of \( E \), we can easily get that for \( \forall X, Y \in H \),
\[ E^*(X + c) = EX + c \cdot |E(X - Y)| \leq E(|X - Y|), \forall X \in H \]
For \( \forall a \in R \), if \( EY = EY \), then \( EX + aY = EY + aEX \).

**Definition 2** Let \( G \subset F \), a function \( V: G \to [0,1] \) is called a capacity if:

1. \( V(\emptyset) = 0 \), \( V(\Omega) = 1 \);
2. \( V(A) \leq V(B) \) for \( \forall A \subseteq B, A, B \in G \).

It is called to be sub-additive if \( V(A \cup B) \leq V(A) + V(B) \) for all \( A, B \in G \) with \( A \cup B \in G \).

Let \( (\Omega, H, E) \) be a sub-linear expectation space and \( E^* \) be the conjugate expectation of \( E \). We denote a pair \( (V, v) \) of capacities by
\[ V(A) = \inf \{ E_\xi : I(A) \leq \xi, \xi \in H \}, v(A) = 1 - V(A^c), \forall A \in F \]

where \( A^c \) is the complement set of \( A \).

It is obvious that \( V \) is sub-additive and
\[ v(A) \leq V(A), \forall A \in F, V(A) = E[I(A)], v(A) = E[I(A)], \forall A \in H \]

If \( f \leq I(A), g, g \in H \), then
\[ Ef \leq V(A) \leq Eg, \hat{E}f \leq v(A) \leq \hat{E}g \]

This implies Markov inequality: \( \forall x \in H, V(|X| \geq x) \leq E[|X|^p] / x^p, \forall x > 0, p > 0 \) from
\[ I(|X| \geq x) \leq |X|^p / x^p, \forall x > 0, p > 0 \]

Next we introduce some important inequalities in sub-linear expectation spaces.
Jensen inequality: \( \left( E[X^r] \right)^{\frac{1}{r}} \leq \left( E[X] \right)^{\frac{1}{r}}, \forall r \in R \)

Hölder inequality: \( E[XY] \leq E \left[ \left( X^p \right)^{\frac{1}{p}} \right] \cdot E \left[ \left( Y^q \right)^{\frac{1}{q}} \right] \), \( \forall X, Y \in H, \frac{1}{p} + \frac{1}{q} = 1 \)

Cauchy inequality: \( E[X_1 + X_2 + \cdots + X_n]^r \leq c_r \left( E[X_1]^r + E[X_2]^r + \cdots + E[X_n]^r \right) \)

where \( c_r = \begin{cases} 1, & 0 < r \leq 1 \\ n^{-r}, & r > 1 \end{cases} \)

**Definition 3** Choquet integral is defined by \( C_B(X) = \int_{0}^{\infty} V(X \geq t) dt + \int_{-\infty}^{0} (V(X \geq t) - 1) dt \)

**Definition 4** (identical distribution) Let \( X_1, X_2 \) be two \( n \)-dimensional random vector defined respectively in sub-linear expectation space \( (\Omega_1, H_1, E_1) \) and \( (\Omega_2, H_2, E_2) \). They are called identically distributed, denoted by \( X_1 = X_2 \), if \( E_1[\phi(X_1)] = E_2[\phi(X_2)] \), \( \forall \phi \in C_{1,Lip}(R^n) \). A sequence \( \{X_n, n \geq 1\} \) of random variable is said to identically distributed if \( X_i = X_1 \), for each \( i \geq 1 \).

**Definition 5** (independence) In a sub-linear expectation space \( (\Omega, H, E) \), a random vector \( Y = (Y_1, Y_2, \cdots, Y_n), Y_i \in H \) is said to be independent to another random vector \( X = (X_1, X_2, \cdots, X_n), X_i \in H \) under \( E \) if for each test function \( \phi \in C_{1,Lip}(R^n \times R^n) \), we have \( E[\phi(X, Y)] = E[E(\phi(X,Y)) | X = x] \), where \( \phi(x) = E(\phi(x,Y)) < \infty \) for all \( x \) and \( E(\phi(x)) < \infty \).

**Definition 6** (Negative dependence) In a sub-linear expectation space \( (\Omega, H, E) \), a random vector \( Y = (Y_1, Y_2, \cdots, Y_n), Y_i \in H \) is said to be negatively dependent to another random vector \( X = (X_1, X_2, \cdots, X_n), X_i \in H \) under \( E \) if for each pair of test function \( \phi_1 \in C_{1,Lip}(R^n) \) and \( \phi_2 \in C_{1,Lip}(R^n) \), we have \( E[\phi_1(X)\phi_2(Y)] \leq E[\phi_1(X)]E[\phi_2(Y)] \), whenever either \( \phi_1 \) and \( \phi_2 \) are coordinate-wise non decreasing or \( \phi_1 \) and \( \phi_2 \) are coordinate-wise non increasing with \( \phi_1(X) \geq 0 \), \( E[\phi_2(Y)] \geq 0 \), \( E[\phi_1(X)\phi_2(Y)] < \infty \), \( E[\phi_1(X)] < \infty \), \( E[\phi_2(Y)] < \infty \).

**Definition 7** (Negative dependence random variables) Let \( \{X_n, n \geq 1\} \) be a sequence of random variables in the sub-linear expectation space \( (\Omega, H, E) \). \( X_1, X_2, \cdots \) are said to negatively dependent if \( X_{i+1} \) is negatively dependent to \( (X_1, X_2, \cdots, X_i) \) for each \( i \geq 1 \).

It is obvious that, if \( \{X_n, n \geq 1\} \) is a sequence of independent random variables and \( f_1(x), f_2(x), \cdots \in C_{1,Lip}(R) \), then \( \{f(X_n), n \geq 1\} \) is also a sequence of independent random variables; if \( \{X_n, n \geq 1\} \) is a sequence of negatively dependent random variables and \( f_1(x), f_2(x), \cdots \in C_{1,Lip}(R) \) are non-decreasing (resp. non-decreasing) function, then \( \{f_n(X_n), n \geq 1\} \) is also a sequence of negatively dependent random variables.
Definition 8 An array of random variables \( \{X_i, 1 \leq i \leq k_n, n \geq 1 \} \) in the sub-linear expectation space \((\mathcal{Q}, H, E)\) is called rowwise negatively dependent random variables if for every given \( n \geq 1 \), \( \{X_i, 1 \leq i \leq k_n \} \) is a sequence of negatively dependent random variables in \((\mathcal{Q}, H, E)\). Where \( \{k_n, n \geq 1 \} \) is a sequence of positive integers and \( k_n \to \infty, n \to \infty \).

4. Main Result

In order to prove our result, we need the following lemma:

**Lemma** Let \( \{X_i, 1 \leq i \leq n, n \geq 1 \} \) is an array of rowwise negatively dependent random variables, \( EX_i = 0 \). Assume that \( \max_{1 \leq i \leq n} |X_i| = O(\log n) \), \( \sum_{i=1}^{n} EX_i^2 = o((\log n)^{-1}) \), then for any \( \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} X_i > \varepsilon \right) < \infty.
\]

**Theorem** Let \( \{X_i, 1 \leq i \leq n, n \geq 1 \} \) is an array of rowwise negatively dependent random variables in the sub-linear expectation space \((\mathcal{Q}, H, E)\). Assume that there exist a random variable \( X \in H \) and a constant \( c \) satisfying

\[
E(h(X_i)) \leq cE(h(X))
\]  
(1)

for all \( n \geq 1, 1 \leq i \leq n \), \( 0 \leq h \in C_{1, \text{Lip}}(R) \).

\[
E(|X|) \leq C_V E(|X|) < \infty
\]  
(2)

Let \( \{a_i, 1 \leq i \leq n, n \geq 1 \} \) is an array of positive real numbers satisfying

\[
\sum_{i=1}^{n} a_i = O(n^{-\gamma})
\]  
(3)

for some \( \gamma > 0 \), and

\[
\sum_{i=1}^{n} a_i^2 EX_i^2 g(a_i X \log n) = o((\log n)^{-1})
\]  
(4)

Then for any \( \varepsilon > 0 \), any positive integer number \( b, b\gamma > 1 \),

\[
\sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} a_i (X_i - EX_i) I \left( \left| a_i X_i \right| \leq \frac{\varepsilon}{b} \right) > \varepsilon \right) < \infty
\]  
(5)

\[
\sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} a_i (X_i - \hat{EX}_i) I \left( \left| a_i X_i \right| \leq \frac{\varepsilon}{b} \right) < -\varepsilon \right) < \infty
\]  
(6)

Proof We replace \( \{X_i, 1 \leq i \leq n, n \geq 1 \} \) in formula (5) with \( \{-X_i, 1 \leq i \leq n, n \geq 1 \} \) to get formula (6). If \( \{X_i, 1 \leq i \leq n, n \geq 1 \} \) is an array of rowwise negatively dependent random variables, then \( \{-X_i, 1 \leq i \leq n, n \geq 1 \} \) is also an array of rowwise negatively dependent random variables. Therefore we just need to prove formula (5). Without losing generality, we assume that \( EX_i = 0 \).

For \( n \geq 1 \), and \( 1 \leq i \leq n \), we define

For n ≥ 1, and 1 ≤ i ≤ n, we define
$$X_m(1) = X_m_1[a_m X_m \leq (\log n)^{-1}] + a_m^{-1}(\log n)^{-1} I(a_m X_m > (\log n)^{-1} - a_m^{-1}(\log n)^{-1}) I(a_m X_m < -(\log n)^{-1}),$$

$$X_m(2) = (X_m - a_m^{-1}(\log n)^{-1}) I((\log n)^{-1} < a_m X_m \leq \frac{\varepsilon}{b},$$

$$X_m(3) = (X_m + a_m^{-1}(\log n)^{-1}) I(-\frac{\varepsilon}{b} \leq a_m X_m < -(\log n)^{-1},$$

$$X_m(4) = -a_m^{-1}(\log n)^{-1} I(a_m X_m > \frac{\varepsilon}{b}) + a_m^{-1}(\log n)^{-1} I(a_m X_m < -\frac{\varepsilon}{b}).$$

It is obvious that $a_m X_m I\left[a_m X_m \leq \left(\frac{\varepsilon}{b}\right)\right] = a_m X_m(1) + a_m X_m(2) + a_m X_m(3) + a_m X_m(4)$.

Therefore we have

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_m X_m I\left[a_m X_m \leq \left(\frac{\varepsilon}{b}\right)\right] > 4\varepsilon\right)$$

$$\leq \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_m X_m(1) > \varepsilon\right) + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_m X_m(2) > \varepsilon\right)$$

$$+ \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_m X_m(3) > \varepsilon\right) + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_m X_m(4) > \varepsilon\right).$$

According to definition 8, we can easily know $\{a_m X_m(1)\}$ is an array of rowwise negatively dependent random variables.

We define a function $g(x) \in C_l Lip(R)$ for $0 < \mu < 1$ as follows: $0 \leq g(x) \leq 1$ for any $x$. When $|x| < \mu$, $g(x) = 1$; when $|x| > 1$, $g(x) = 0$. And then we have

$$I(|x| \leq \mu) \leq g(x) \leq I(|x| \leq 1), I(|x| > 1) \leq 1 - g(x) \leq I(|x| > \mu). \quad (7)$$

According to (7) and $C_r$ inequality, we have

$$E\left(|X_m(1)|^r\right) \leq E\left(|X|^r g(\mu a_m X \log n)\right) + a_m^{-r}(\log n)^{-r} E(1 - g(a_m X \log n))$$

$$\leq E\left(|X|^r g(\mu a_m X \log n)\right) + a_m^{-r}(\log n)^{-r} V(|a_m X| > \mu(\log n)^{-1}) \quad (8)$$

Next we above prove $\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_m X_m(1) > \varepsilon\right) < \infty$.

From Markov inequality and formula (1)(2)(3)(4)(8), we have

$$\sum_{i=1}^{n} E(a_m X_m(1) - EX_m(1))^2 \leq \sum_{i=1}^{n} a_m^2 E(2X_m^2(1) + 2(\text{EX}_m(1))^2)$$

$$\leq cE(X_m(1))^2 \leq \sum_{i=1}^{n} a_m^2 EX^2 g(\mu a_m X \log n) + (\log n)^{-r} \sum_{i=1}^{n} V(|a_m X| > \mu(\log n)^{-1})$$

$$\leq o(\log n)^{-1} + (\log n)^{-r} \sum_{i=1}^{n} a_m E(|X|)$$

$$\leq o((\log n)^{-1}) + (\log n)^{-2} O(n^{-r}) C_{\nu}(|X|) = o((\log n)^{-1}).$$

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Therefore from lemma we have \[ \sum_{n=1}^{\infty} V\left(\sum_{i=1}^{n} a_m (X_n (1) - EX_n (1)) \geq \frac{\varepsilon}{2}\right) < \infty. \]

Because if \[ \sum_{i=1}^{n} a_m EX_n (1) \rightarrow 0, n \rightarrow \infty \] is proved, then \[ \sum_{n=1}^{\infty} V\left(\sum_{i=1}^{n} a_m X_n (1) > \varepsilon\right) < \infty \] is proved.

Next we prove \[ \sum_{i=1}^{n} a_m EX_n (1) \rightarrow 0, n \rightarrow \infty. \]

Thus \[ \sum_{n=1}^{\infty} V\left(\sum_{i=1}^{n} a_m X_n (1) > \varepsilon\right) < \infty \] is proved.

Next we prove \[ \sum_{n=1}^{\infty} V\left(\sum_{i=1}^{n} a_m X_n (2) > \varepsilon\right) < \infty. \]

From \[ X_n (2) = (X_n - a_m^{-1} (\log n)^{-1})I((\log n)^{-1} < a_mX_n \leq \frac{\varepsilon}{b}) \]

we can obtain \[ 0 \leq a_mX_n (2) \leq \frac{\varepsilon}{b}, \quad \left| \sum_{i=1}^{n} a_m X_n (2) \right| = \sum_{i=1}^{n} a_m X_n (2) > \varepsilon. \] It means that there exists at least one positive integer so that \( X_n (2) \neq 0 \).

Therefore, from Markov inequality and formula (1)-(4), we can obtain

\[ V\left(\sum_{i=1}^{n} a_m X_n (2) > \varepsilon\right) \leq \sum_{1<k_1<\ldots<k_n \leq n} V\left(|a_{k_1} X_{k_1}| \geq (\log n)^{-1}, \ldots, |a_{k_n} X_{k_n}| \geq (\log n)^{-1}\right) \]

\[ << \sum_{1<k_1<\ldots<k_n \leq n} E\left((1-g(a_{k_1} X \log n)) \cdots (1-g(a_{k_n} X \log n))\right) \]

\[ \leq \sum_{1<k_1<\ldots<k_n \leq n} E(1-g(a_{k_1} X \log n)) \cdots E(1-g(a_{k_n} X \log n)) \]

\[ \leq \left(\sum_{i=1}^{n} E(1-g(a_{m} X \log n))\right)^b \]
\[ \leq \left( \sum_{i=1}^{n} V(\mathbf{a}_m X_i) > \mu(\log n)^{-1}) \right)^b \]

\[ \ll \left( \log n \sum_{i=1}^{n} E(|X|) \right)^b \]

\[ \ll \left( \log n(n^{-\gamma}) E(|X|) \right)^b \]

\[ \leq c((\log n)(n^{-\gamma}))^b. \]

From \( \gamma b > 1 \), \( \sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} \mathbf{a}_m X_m(2) > \varepsilon \right) < \infty \) is proved.

By the same methods as the \( \sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} \mathbf{a}_m X_m(2) > \varepsilon \right) < \infty \), we can get

\[ \sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} \mathbf{a}_m X_m(3) > \varepsilon \right) < \infty. \]

Finally we prove \( \sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} \mathbf{a}_m X_m(4) > \varepsilon \right) < \infty. \)

From

\[ X_m(4) = -\mathbf{a}_m^{-1}(\log n)^{-1} I(\mathbf{a}_m X_m > \varepsilon) b + \mathbf{a}_m^{-1}(\log n)^{-1} I(\mathbf{a}_m X_m < \varepsilon), \]

we obtain

\[ \mathbf{a}_m X_m(4) \leq (\log n)^{-1} I(\mathbf{a}_m X_m > \varepsilon) b \leq \varepsilon \]

Thus there are at least \( b \) subscripts \( i \) so that

\[ |\mathbf{a}_m X_m| > \varepsilon \]

Therefore, from Markov inequality, formula (1)-(3) and (6), for \( \gamma b > 1 \), we can obtain

\[ \sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} \mathbf{a}_m X_m(4) > \varepsilon \right) \ll \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} E(1 - g(\mathbf{a}_m X / \varepsilon)) \right)^b \]

\[ \leq \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} V(\mathbf{a}_m X > \varepsilon \mu / \varepsilon) \right)^b \]

\[ \leq \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} \mathbf{a}_m E|X|(|\varepsilon \mu / \varepsilon)|^{-1} \right)^b \]

\[ \ll \sum_{n=1}^{\infty} \left( E|X|O(n^{-\gamma}) \right)^b < \infty. \]

Therefore \( \sum_{n=1}^{\infty} V \left( \sum_{i=1}^{n} \mathbf{a}_m X_m(4) > \varepsilon \right) < \infty. \)

So far, the theorem is proved.
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