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Monotone bounded theorem on the hyperbolic plane

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Abstract: *Hyperbolic numbers have a similar structure to complex number, which are a generalization of real numbers, but they are is an exchangeable ring containing a zero factor. The purpose of this article is to study monotone bounded theorems on the hyperbolic plane. This article breaks the limitations of previous discriminant conditions for the convergence of hyperbolic series, and propose a simple but effective of new method that provides a new discriminant condition of it. This paper provides some theoretical basis for the study of hyperboloid properties, which have a wide application background in mathematical analysis and physics. For example, in the field of mechanical and advanced engineering informatics, dynamic elastic analysis is performed using hyperbolic roofs.*

Keywords: *Hyperbolic Number, Monotone Bounded Theorem, Hyperbolic Modulus*

1. Introduction

In 2009, Zhang Shi-qin and Guo Bai-ni present monotonicity results of a function involving the inverse hyperbolic sine, and they obtain some lower bounds for the inverse hyperbolic sine.[1] In 2007, Zhang Wei-quan discuss a matrix approach to the representation of hyperbolic numbers.[2] Besides, other researchers used hyperbolic imaginary unit of Clifford algebra to introduce the concepts of hyperbolic number and hyperbolic complex plane.

Additionally, Zhao Lili construct the proper order relation of points on the coordinate plane, and give extended the monotone bounded principle to the coordinate plane. Besides, others study the general methods to determine the limit of recurrent sequences and give some concrete examples to illustrate the method, including using the monotone bounded theorem to determine convergence of recursive series.[3]

The accurate researches on the hyperbolic numbers are of great significance for the physical measurement, water conservancy engineering, and construction industries. In current study, researchers use the refractive index to determine a hyperboloidal concave acoustic lens which can improve the minimized region of the manipulation capability to designed H-FAV.[4] In order to design gear tooth of manufacturing and analysis on an integral system, some researchers develop hypoid gear as well as others showed the effective and feasible of using technology of concrete double curvature arch dam joints for secondary grouting.[5]

The purpose of this article is to determine a new method that provides a simple but powerful discriminant condition for determining the convergence of hyperbolic series. And we end up with a monotone bounded theorem for hyperbolic series, which is important for solving practical problems and deriving relevant mathematical theorems. First, we give some fundamental property, definitions, and geometric interpretations about hyperbolic numbers and hyperbolic plane. Next, hyperbolic monotone bounded theorem is given. At last, we mention a number of backgrounds with factual applications and potential applications of this property, including mechanical engineering, optics, and quantum mechanics.

2. The basic fundamental of the hyperbolic numbers

2.1 The structure of the hyperbolic numbers

There are many theorems on the set of real numbers, and the set of hyperbolic numbers is no exception. We first give the definition of the set of hyperbolic numbers:

$$\mathbb{D}:=\{\xi=x+\boldsymbol{k}y: x,y\in\mathbb{R}\} \tag{1}$$

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and the hyperbolic unit $\mathbf{k}$ satisfies $\mathbf{k}^2 = 1$ and $\mathbf{k} \neq \pm 1$. In some literature, it is known by many other terms like double, spacetime, perplex or split complex numbers.

The set $\mathbb{D}$ is a commutative ring, which has several the operations of addition and multiplication. These are as follows:

$$\begin{aligned} \xi_1 + \xi_2 &= (x_1 + \mathbf{k}y_1) + (x_2 + \mathbf{k}y_2) \\ &= (x_1 + x_2) + (y_1 + y_2)\mathbf{k} \end{aligned} \tag{2}$$

and

$$\begin{aligned} \xi_1\xi_2 &= (x_1 + \mathbf{k}y_1)(x_2 + \mathbf{k}y_2) \\ &= (x_1x_2 + y_1y_2) + \mathbf{k}(x_1y_2 + x_2y_1) \end{aligned} \tag{3}$$

Where $\xi_1 = (x_1 + \mathbf{k}y_1)$ and $\xi_2 = (x_2 + \mathbf{k}y_2)$. And for any $\xi = x + \mathbf{k}y \in \mathbb{D}$, the real part of ξ is defined as $\mathrm{Re}(\xi) = x$, the hyperbolic part is defined as $\mathrm{Im}(\xi) = y$ as well.

The notice thing is the conjugate of ξ called $\overline{\xi}$, and we define $\overline{\xi}$ as $\overline{\xi} = x - \mathbf{k}y$. Thus, hyperbolic number can be written as follows:

$$e := \frac{1+\mathbf{k}}{2} \text{ and } e^+ := \frac{1-\mathbf{k}}{2} \tag{4}$$

The following formulas and conclusions of the properties of the number multiplication operation can be referred from the above definitions:

$$\begin{aligned} ee^+ &= \frac{1+k}{2} \cdot \frac{1-k}{2} \\ &= \frac{1-k^2}{4} = 0 \end{aligned} \tag{5}$$

Where $\{e, e^+\}$ is called the idempotent base of $\mathbb{D}$. Besides, it is worth noting that the ring of hyperbolic number has zero-divisors, which are idempotent elements with the following specific formulas:

$$\begin{aligned} e^2 &= \frac{(1+\mathbf{k})}{2} \cdot \frac{(1+\mathbf{k})}{2} \\ &= \frac{1+2k+\mathbf{k}^2}{4} = e \end{aligned} \tag{6}$$

For e^+, it could derive the same equation by the same method equivalently:

$$\begin{aligned} (e^+)^2 &= \frac{(1-\mathbf{k})}{2} \cdot \frac{(1-\mathbf{k})}{2} \\ &= \frac{1-2k+\mathbf{k}^2}{4} = e^+ \end{aligned} \tag{7}$$

Additionally, for e and e^+, we can derive properties of basic numerical operations:

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$$e+e^{+}=\frac{1+\mathbf{k}}{2}+\frac{1-\mathbf{k}}{2}=1$$
$$e-e^{+}=\frac{1+\mathbf{k}}{2}-\frac{1-\mathbf{k}}{2}=\mathbf{k} \tag{8}$$

The conclusion are

$$e+e^{+}=1 \text{ and } e-e^{+}=\mathbf{k} \tag{9}$$

All above are idempotent zero divisors forming the whole ring. To further illustrate, the idempotent representation is

$$\xi=(x+y)e+(x-y)e^{+}=:ae+be^{+} \tag{10}$$

Where $\xi=x+\mathbf{k}y\in\mathbb{D}$, and $a=(x+y),b=(x-y)$. According to the basic arithmetic properties of algebra, we know that the zero-divisors in the set $\mathbb{D}$ are real multiples of e and e^{+}. Thus, given some hyperbolic numbers easy a series of conclusions of $\xi_1=a_1e+a_2e^{+}$ and $\xi_2=b_1e+b_2e^{+}$ as follows:

$$\xi_1+\xi_2=(a_1+b_1)e+(a_2+b_2)e^{+}$$
$$\xi_1\xi_2=(a_1b_1)e+(a_2b_2)e^{+} \tag{11}$$

The set

$$\mathbb{D}^{+}:=\left\{\xi_1=ae+be^{+}:a\geq 0,b\geq 0\right\} \tag{12}$$

is called the set of non-negative hyperbolic numbers. Or equivalently:

$$\mathbb{D}^{-}:=\left\{\xi_1=ae+be^{+}:a\leq 0,b\leq 0\right\} \tag{13}$$

is called the set of non-positive hyperbolic numbers. As a result of that $\xi_1,\xi_2\in\mathbb{D}^{+}$, we can conclude that $\xi_1\xi_2\in\mathbb{D}^{+}$. Thus, according to closed interval set theorem, $\mathbb{D}^{+}$ is closed under the multiplication.(Fig 1)

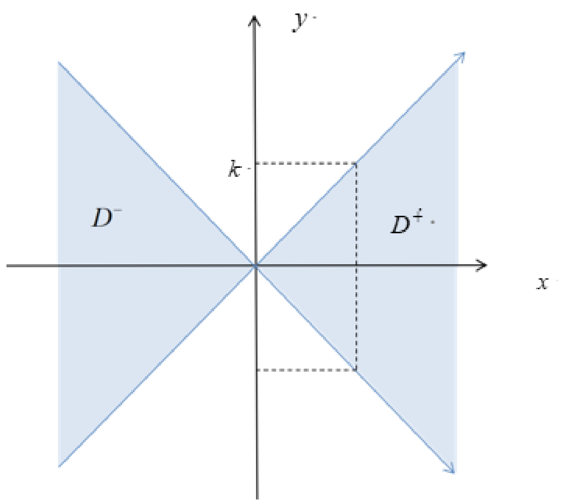

Figure 1: The positive and negative hyperbolic numbers

Thus, if $\xi,\omega\in\mathbb{D}$, we have

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$$\xi \preceq \omega \iff \omega - \xi \in \mathbb{D}^{+} \tag{14}$$

And it have that ω is $\mathbb{D}$ -greater than ξ, or

$$\omega \preceq \xi \; i \iff \omega - \xi \in \mathbb{D}^{-} \tag{15}$$

and it have that ξ is $\mathbb{D}$ -greater than ω. In conclusion, we say that $\xi \in \mathbb{D}^{+}$ is equivalent to $\xi \succ 0$ and for $\xi \in \mathbb{D}^{+} - \{0\}$, it is equivalent to $\xi \succ 0$. Equivalently, we say that $\xi \in \mathbb{D}^{-}$ is equivalent to $\xi \prec 0$ and for $\xi \in \mathbb{D}^{-} - \{0\}$, it is equivalent to $\xi \prec 0$. For ξ_1, ξ_2, the relationship between them is reflexive, transitive and antisymmetric, which donotes a partial order in $\mathbb{D}$. Therefore, we inject real line embedding in D by $\varphi: \mathbb{R} \to \mathbb{D}$, if $a \in R$, we have

$$\varphi(a) = \tilde{a} = ae + ae^{+} \tag{16}$$

Definition 1 For hyperbolic numbers, $\xi = a_1 e + b_1 e^{+}$ and $\omega = a_2 e + b_2 e^{+}$ in $\mathbb{D}$, we define closed hyperbolic interval $[\xi, \omega]_{\mathbb{D}}$ that

$$[\xi, \omega]_{\mathbb{D}} = \{u \in \mathbb{D} : \xi \preceq u \preceq \omega\} \tag{17}$$

Where $u = u_1 e + u_2 e^{+} \in [\xi, \omega]_{\mathbb{D}}$, additionally

$$a_1 \le u_1 \le a_2 \text{ and } b_1 \le u_2 \le b_2 \tag{18}$$

We have the following explanation for degeneracy and non-degeneracy. There is a hyperbolic number $\omega - \xi$, which is a non-negative zero divisor, and we define that the hyperbolic interval $[\xi, \omega]_{\mathbb{D}}$ is degenerate. Also, if there is an invertible positive hyperbolic number $\omega - \xi$, and then we define that the hyperbolic interval $[\xi, \omega]_{\mathbb{D}}$ is non-degenerate. A hyperbolic interval is an interval consisting of two hyperbolic numbers, which is similar to a real number interval. The concept of length of a hyperbolic interval is the same as that of a real interval, i.e., it is the distance between two endpoints. It's worth noting that the length of any hyperbolic interval is a non-negative hyperbolic number.

in the hyperbolic plane, any number $\xi = x + ky =: ae + be^{+}$ could respectively be given by the point $(x, y) \in \mathbb{R}^2$ or point $(a, b) \in \mathbb{R}^2$. We can use either the standard base or the idempotent base to define the modulus of hyperbolic numbers. In many other previous studies, the modulus is a positive hyperbolic number, meaning that it is a non-negative quantity that belongs to the set of hyperbolic numbers. While in this study, we consider that the modulus is a positive real number. In other words, the norm utilized in this study is the norm defined in accordance with the standard base of hyperbolic numbers. This norm provides a measure of the size or magnitude of hyperbolic numbers within the framework of the standard base.

Definition 2 For hyperbolic number $\xi = ae + be^{+}$, the positive hyperbolic number

$$|\xi|_k = |ae + be^{+}| = |a| e + |b| e^{+} \tag{19}$$

is defined by the hyperbolic modulus of ξ. (Fig 2)

Since $|\xi|_k$ fulfills the properties of modulus, for any $\xi, \omega \in \mathbb{D}$,

1) $|\xi|_k = 0 \Leftrightarrow \xi = 0$

2) $|\xi\omega|_k = |\xi|_k |\omega|_k$

3) $|\xi+\omega|_k \preceq |\xi|_k + |\omega|_k$

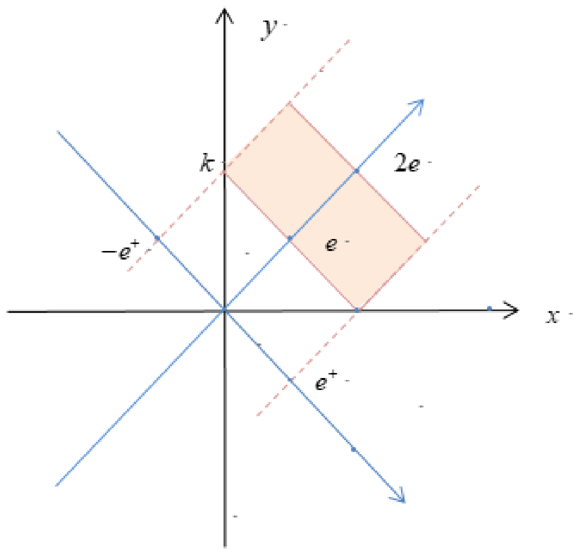

Figure 2: The hyperbolic interval

3. Results

3.1 The establishment of Monotone bounded

Theorem 3.1. (hyperbolic monotone bounded theorem) Let $\{\xi_n\}$ be a sequence of hyperbolic numbers. If $\xi_n \preceq \xi_{n+1}$ for $n=1,2,\cdots$, there exists an integer $M>0$, since for any $\boldsymbol{n} \in \mathbb{N}^+$ it holds that $|\xi_n|_k \preceq M$, ξ_n converges to a limit.

Proof For every integer $\boldsymbol{n} \in \mathbb{N}^+$, there is a sequence $\{\xi_n\}$ of points,

$$\xi_n = \xi_{1,n}e + \xi_{2,n}e^+$$
$$\xi_{n+1} = \xi_{1,n+1}e + \xi_{2,n+1}e^+ \tag{20}$$

Since the concept of partial order: $\xi_n \preceq \xi_{n+1}$,hence

$$\xi_{1,n}e + \xi_{2,n}e^+ \preceq \xi_{1,n+1}e + \xi_{2,n+1}e^+ \tag{21}$$

Since the partial order is less than or equal to, which illustrate that their difference is in $\mathbb{D}^+$. The corresponding coordinates are also consistent with the relationship that the partial order is less than or equal to, we get

$$\begin{cases} \xi_{1,n} \le \xi_{1,n+1} \\ \xi_{2,n} \le \xi_{2,n+1} \end{cases} \tag{22}$$

It means coordinate point decomposition to the axes of e and e^+, which are monotonous.

The proof of bounded is as follows. Since the properties of the mold, $|\xi_n|_k \preceq M$ can be written as:

$$\left|\xi_n\right|_k = \left|\xi_{1,n}\right| e + \left|\xi_{2,n}\right| e^{+} \preceq M = Me + Me^{+} \tag{23}$$

Therefore, we obtain:

$$\begin{cases} \left|\xi_{1,n}\right| \le M \\ \left|\xi_{2,n}\right| \le M \end{cases} \tag{24}$$

It is easy to see that the decomposition onto the two real axes is still monotonically bounded.

Therefore, we conclude that $\xi_{1,n}$ and $\xi_{2,n}$ converges to a limit, which means that ξ_n converges to a limit.(Fig 3)

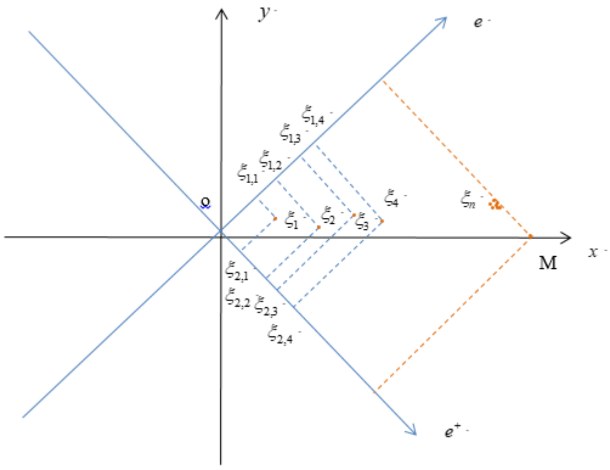

Figure 3: Geometric meaning of monotone bounded

4. Conclusions

Hyperbolic numbers share a structure with complex numbers, which are a generalization of real numbers, but they form an exchangeable ring with a zero factor. This article examine monotone bounded theorems on the hyperbolic plane and present a convergence discrimination method for hyperbolic series. Theoretical foundations provided by these findings enable the exploration of the properties of the hyperbolic plane, which have diverse practical applications such as utilizing hyperbolic roofs for dynamic elastic analysis, in the fields of mechanism and machine, and in advanced engineering informatics.

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