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Research on Construction of Conference Matrices and Transformation between Conference Matrices and Equiangular Tight Frames

Lingfei Zhang[*1], Yunbo Yang[2], Yilin Ge[3], Yudi Cai[4]

1 Hong Kong Baptist University, Kowloon City, Hksar 999077, China
2 Harbin Institute of Technology (Weihai), Weihai 264209, China
3 Harrisburg Academy, Wormleysburg, PA 17043, United States
4 World Foreign Language Academy, Shanghai 200032, China
** Corresponding author email: 17251699@life.hkbu.edu.hk*

ABSTRACT. *Equiangular tight frames arise in various applications in communications and coding. ETFs have close connections with such combinatorial objects as strongly regular graphs, difference sets and Steiner systems. This paper demonstrates with examples one way of constructing a special kind of matrices called conference matrices while showing the limitation of this method, and also demonstrates with examples the correspondence between conference matrices and d-by-2d ETFs.*

KEYWORDS: *Equiangular tight frame, Conference matrix, Gram matrix*

1. Introduction

The objects this paper focuses on are *conference matrices* and *equiangular tight frames*. The former is a type of matrices consisting of only -1, 0, and 1, and the latter can be seen as sets of equiangular vectors through origins, with the angle between any two distinct vectors maximized [1].

The term *frame* refers to a generalization of basis vectors of an inner product space. A frame may not be linearly independent, thus in signal processing, it helps represent a signal redundantly, unlike the case where basis vectors are used so that only one way of representation is allowed. An ETF is also named an *optimal Grassmannian frames* [2], or a *2-uniform frame* [3].

Definition 1. Suppose $S = [\, f_1\ f_2\ \dots\ f_N\,]$ is a d-by-N matrix, then S is called an *equiangular tight frame* if

(1) Every column of S has unit norm: $\| f_i \| = 1$ for i = 1, …, N

(2) The inner products between any two different columns, when taken absolute value, are some constant: $|\langle f_i, f_j\rangle| = \alpha$ for some α for i, j = 1, …, N and i $\neq$ j

(3) $SS^* = (N/d)\, I$ where S^* is the conjugate transpose of S.

Though ETFs may become valuable in areas such as communication and sparse approximation [2] [1] [4], the existence of ETFs proves to be sporadic. For most (d, N) pairs an ETF does not exist. Research in the past has been concentrated on such fields, e.g., finding necessary conditions on (d, N) pairs so that an ETF may exist [5] [1], and to construct new ETFs numerically [6].

Among tries to construct ETFs, one way is referring to conference matrices in order to get (d, 2d) ETFs. However, the prerequisite is that the corresponding conference matrices are known. Researchers are still developing methods for finding new conference matrices [7].

A method by Paley is stated in [8]. This paper demonstrates this method of construction.

Moreover, this paper finds the limitation of this method that for fields with more than one zero squares such construction may fail. This paper also demonstrates with examples the transformation of 2d-order conference matrices and d-by-2d ETFs.

In addition, we may have found an error of an equation in [9].

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2. Basics of Etfs and Conference Matrices

Theorem 2. If $S = [\, f_1\ f_2 \ldots f_N \,]$ is a d-by-N matrix with unit norm columns, then S is an ETF if and only if

$|\langle f_i, f_j \rangle| = \frac{\sqrt{N-d}}{\sqrt{d(N-1)}}$ for i, j = 1, …, N and i $\neq$ j

In fact, if S, with unit norm columns is not an ETF then $|\langle f_i, f_j \rangle| \geq \frac{\sqrt{N-d}}{\sqrt{d(N-1)}}$. This result gives a concrete value for α in Definition 1. See [5] for the proof. This result also gives a concrete value for the cosine of "angle between any two vectors" in the introduction part.

Definition 3. An n-by-n matrix C is called a *conference matrix of order n* if

(1) The diagonal elements of C are zeros, and the off-diagonal elements are +1 or -1.

(2) $CC^T = (n-1)\, I$.

Note that n must be even. By definition, for two different rows $C_{i,:}$ and $C_{j,:}$, their inner product is zero: $0 = C_{i,1}\, C_{j,1} + \ldots + C_{i,n}\, C_{j,n}$. The right-hand side have two zero terms because some diagonal elements $C_{i,a}$ and $C_{j,b}$ (a, b$\in \{1, \ldots, n\}$) are zeros. Since terms other than the two zeros are either +1 or -1, (n-2) must be even so that the summation of the remaining (n-2) terms is 0.

If $n \equiv 2\ (mod\ 4)$, then C is said to be *symmetric*. If $n \equiv 0\ (mod\ 4)$, then C is *skew-symmetric*. Terms *symmetric* and *skew-symmetric* are actually referring to the submatrix after removing the 1st row and column of C [8]. In fact, symmetric and skew-symmetric matrices have close connections with real and complex d-by-2d ETFs, respectively.

3. Constructing Conference Matrices Using Paley's Method

The idea by Paley is stated in [8]. Here we demonstrate the method with examples. Note that $5 = 1^2 + 2^2$ and 5 is also a prime power, hence we can try to construct a conference matrix C with order 6.

First set $C_{1,:}$ and $C_{:,1}$ to be 1, except for their intersection which is 0. Trim off $C_{1,:}$ and $C_{:,1}$ and consider the remaining 5-by-5 matrix M. Consider 5-element field {0, 1, 2, 3, 4} with addition and multiplication (mod 5).

Now comes Paley's construction: Consider column $M_{:,1}$. If (a-1) is a non-zero square in the field, i.e., if (a-1) = 1 or 4 in this case, then set $M_{a,1} = +1$. If (a-1) is not zero nor a square, set $M_{a,1} = -1$. Set $M_{1,1} = 0$.

The 2nd columns $M_{:,2}$ is built by shifting down by one unit every element of $M_{:,1}$, $M_{:,3}$ by shifting down by one unit every element of $M_{:,2}$, and so on. Then we get:

$$\begin{matrix} 0 & +1 & -1 & -1 & +1 \\ +1 & 0 & +1 & -1 & -1 \\ -1 & +1 & 0 & +1 & -1 \\ -1 & -1 & +1 & 0 & +1 \\ +1 & -1 & -1 & +1 & 0 \end{matrix}$$

Augment M by the removed row and column $C_{1,:}$ and $C_{:,1}$. It is easy to check by direct calculation that we get a conference matrix:

$$\begin{matrix} 0 & +1 & +1 & +1 & +1 & +1 \\ +1 & 0 & +1 & -1 & -1 & +1 \\ +1 & +1 & 0 & +1 & -1 & -1 \\ +1 & -1 & +1 & 0 & +1 & -1 \\ +1 & -1 & -1 & +1 & 0 & +1 \\ +1 & +1 & -1 & -1 & +1 & 0 \end{matrix}$$

For another example, with the filed {0, 1, 2, 3, 4, 5, 6} and addition and multiplication (mod 7), we can get the 7-by-7 M. Notice now {1, 2, 4} are non-zero squares and so the 2nd, 3rd and 5th elements of the first column is +1:

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$$\begin{matrix} 0 & -1 & -1 & +1 & -1 & +1 & +1 \\ +1 & 0 & -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & 0 & -1 & -1 & +1 & -1 \\ -1 & +1 & +1 & 0 & -1 & -1 & +1 \\ +1 & -1 & +1 & +1 & 0 & -1 & -1 \\ -1 & +1 & -1 & +1 & +1 & 0 & -1 \\ -1 & -1 & +1 & -1 & +1 & +1 & 0 \end{matrix}$$

It can be verified that the corresponding 8-by-8 C satisfies $CC^T = 7I$.

The third example is the 17-element field based on {0, 1, …, 16}. The squares of this field is {1, 2, 4, 8, 9, 13, 15, 16}. Therefore, $M_{2,1}$, $M_{3,1}$, $M_{5,1}$, $M_{9,1}$, $M_{10,1}$, $M_{14,1}$, $M_{16,1}$, $M_{17,1}$ are assigned with 1. After augmentation with the all-one row and all-one column back, we get an 18-by-18 conference matrix.

4. Limitations of the Above Method

Though Paley has shown the existence of n-by-n symmetric conference matrices for all $n = p^a + 1$ where p is a prime number and a is a natural number [2], just the above method is not enough to construct all such conference matrices, as this method is not applicable for all finite fields whose orders are such p^a. This appears when more than one element's square is zero. For example, the square of 3 and 6 of the field {0, 1, …, 8} are both zero, and the above method fails in constructing a corresponding 10-by-10 conference matrix.

If we follow the aforementioned construction, the 1st column of matrix M should be $[0\ 1\ {-1}\ {-1}\ 1\ {-1}\ {-1}\ 1\ {-1}]^T$. However, under such construction the 10-by-10 matrix A does not satisfy $AA^T = 9I$.

For constructing a 10-by-10 conference matrix, see [10].

5. Transformation between Conference Matrices and d-by-2d Etfs

Definition 4. A matrix R is called a *Hermitian matrix* if $R = R^*$

The following Lemma is proved in [2], which is important for getting ETFs from conference matrices.

Lemma 5. Let d, N $\in \mathbb{N}$ with N $\geq$ d. Assume R is a Hermitian N $\times$ N matrix with entries $R_{i,i} = 1$ and

$|R_{i,j}| = \frac{\sqrt{N-d}}{\sqrt{d(N-1)}}$ if R is real

$|R_{i,j}| = i\frac{\sqrt{N-d}}{\sqrt{d(N-1)}}$ if R is complex

for i, j = 1, …, N, $i \neq j$.

If the eigenvalues $\lambda_1, \ldots, \lambda_N$ of R are such that

$\lambda_1 = \ldots = \lambda_d = N / d$ and $\lambda_{d+1} = \ldots = \lambda_N = 0$

then there exists an ETF.

More specifically, it is pointed out in [2] that the ETF can be extracted with singular value decomposition: suppose $R = SVD$ is a singular value decomposition of R, and suppose the non-zero eigenvalues are the first d elements on the diagonal of V, then $f_k = \frac{\sqrt{N}}{\sqrt{d}}\ \{V_{k,l}\}_{l=1}^{d}$ for k = 1, …, N form an ETF.

Recall that theorem 2 tells for an ETF $S = [\, f_1\ f_2 \ldots f_{2d}]$ where $f_i \in \mathbb{R}^d$, i = 1, 2, …, 2d,

$$\alpha = \frac{1}{\sqrt{(2d-1)}}$$

By Lemma 5, to get a d-by-2d ETF, use a symmetric conference matrix C of order 2d, and compute the Hermitian matrix $R = \frac{1}{\sqrt{(2d-1)}}\ C + I$, while to get a complex ETF, use a skew-symmetric C and compute $R = i\frac{1}{\sqrt{(2d-1)}}C + I$ [9]. We then calculate the spectral decomposition and check if the eigenvalues are 0 and 2 with multiplicity d for both. An ETF can then be extracted if this is the case.

Conversely, given a d-by-2d ETF a conference matrix can be built.

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Definition 6. The *Gram matrix* G for a set of vectors $S = [\ f_1\ f_2 \dots f_N\ ]$ is given by

$G = S^T S$, if S is real

$G = S^* S$, if S is complex

In other words, a Gram matrix is a Hermitian matrix of inner products.

According to [2], If S is real then the Gram matrix G for a d-by-2d equiangular tight frame satisfies $|\ G_{i,j}\ | = \alpha$ and $G_{i,I} = 1$ for i, j = 1, 2, …, N. Then $C = \frac{1}{\alpha}(G - I)$ is a symmetric conference matrix. The case where S is complex is similar. Note that in [5], such formula is defined for every ETF instead of only the 2-by-2d ones, which is named *signature matrices* of ETFs.

For example, consider the 3-by-6 real ETF:

$$\frac{1}{\sqrt{1+\varphi^2}}\begin{matrix} 0 & 0 & 1 & 1 & \varphi & -\varphi \\ 1 & 1 & \varphi & -\varphi & 0 & 0 \\ \varphi & -\varphi & 0 & 0 & 1 & 1 \end{matrix}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ with $\alpha = \frac{1}{\sqrt{5}}$. Then $\frac{1}{\alpha}(G - I)$ gives a conference matrix

$$\begin{matrix} 0 & -1 & 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & -1 & 0 \end{matrix}$$

Note that in [9] the former formula is given as $R = \sqrt{(2d-1)}\ C + I$ instead of $\frac{1}{\sqrt{(2d-1)}}\ C + I$, which is likely to be a typo, since if we are to reversely get the Gram matrix G from C. The equation should be $G = \alpha C + I = \frac{1}{\sqrt{2d-1}}\ C + I$. Moreover, if we use the above matrix to undergo singular value decomposition of the matrix given by $\sqrt{5}\ C + I$, it turns out that the eigenvalues are six and four, rather than zero and two, which are required by Lemma 5.

Also note that by van Lint and Seidel, if a symmetric conference matrix of order n exists, then (n-1) is the sum of two squares, and therefore there is no such matrix of order 22 or 34 ([8]). This is in line with the fact that no 11-by-22 and 17-by-34 real ETFs exist (See [11] for lists of dimensions of known non-trivial ETFs). Actually, d-by-2d ETFs correspond to a type of strongly regular graphs called conference graphs [11].

6. Conclusion

Equiangular tight frames have potential applications in different areas. One of the methods used to get d-by-2d ETFs is through conference matrices. We demonstrate one kind of construction for conference matrices with examples, and also find the limitation of such construction. Further, we demonstrate the process of transformation between conference matrices and equiangular tight frames, while finding an error of an expression in [9]. This paper may give a first insight to those who are to study d-by-2d ETFs.

It remains for us to study whether Paley’s construction mentioned in [8] fails for every field with more than one zero-square. If it does not, under what cases will it work and give the desired conference matrix.

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